Neighborhood Structure of Rotations

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1 Introduction

The basic idea of the registration approach presented in [2] is to calculate for each point in range image 1 the rotation and translation to each point in range image 2 with the help of their local frames[1] and to store the resulting transformation parameters $(\alpha, \beta, \gamma, t_x, t_y, t_z)$ in a six dimensional Hough table by incrementing a counter in the table at position $(\alpha, \beta, \gamma, t_x, t_y, t_z)$. Since transformations calculated from correct point correspondences result in the same transformation while all other transformations are distributed more or less randomly in the parameter space, a peak is expected in the Hough table at the position of the searched transformation.

Since the number of Hough table cells is limited due to limited memory resources, each Hough table cell has only a limited resolution $(\Delta\alpha, \Delta\beta, \Delta\gamma, \Delta t_x, \Delta t_y, \Delta t_z)$ that is connected to the size of the initial parameter domain. This Hough table cell resolution is far beyond the desired accuracy of the transformation determination when no or only little[2] a priori knowledge is given for the initial parameter domain. The solution to this problem is to start with a rough resolution for each Hough table cell, detect the Hough peak so that the parameter domain can be restricted to the neighborhood of the peak, and then iterate the procedure by choosing each time a higher resolution for the Hough table cells.

There are some practical problems in determining the neighborhood of the transformation that corresponds to the Hough peak. Since the topology of rotation

[1] A local frame in a surface point can be given e.g. by the normal and the directions of minimal and maximal curvature in this point.

[2] The translation domain can be easily reduced by an initial translation overlaying the centers of mass of the two given range images and then only considering all reasonable remaining translations.

parameters is non trivial (independent of the chosen parameterization) Hough cells that are far away from each other in the Hough table can be close to each other in the sense of corresponding rotations. A simple example is a rotation angle φ whose values $\varphi_1 = 0+\varepsilon$ and $\varphi_2 = 2\pi-\varepsilon$ are close to each other in the sense of rotations but far away in the Hough table since they lie on opposite sites of the table.

In this paper we examine the neighborhood structure for three rotation representations: the Euler angle representation, the axis and angle representation and the quaternion representation.

2 Euler angle neighborhood structure

2.1 Definition of Euler angles

We use the Euler angle definition of [4]. A general rotation matrix in this representation is given by the following matrix product:

$$\begin{aligned}\mathbf{R}(\alpha,\beta,\gamma) &= \begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{pmatrix}\begin{pmatrix}\cos\gamma & -\sin\gamma & 0\\ \sin\gamma & \cos\gamma & 0\\ 0 & 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}\cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma-\sin\alpha\cos\gamma & \cos\alpha\sin\beta\\ \sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma+\cos\alpha\cos\gamma & \sin\alpha\sin\beta\\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta\end{pmatrix} \qquad (1)\end{aligned}$$

where

$$0 \le \alpha < 2\pi, \qquad (2)$$

$$0 \le \gamma < 2\pi, \qquad (3)$$

$$0 \le \beta \le \pi. \qquad (4)$$

2.2 Extraction of Euler angles from rotation matrices

Transforming point i of the first image in point j of the second image with the help of the points' local frames results in a translation vector $\boldsymbol{t}$ and especially in a rotation matrix[3] $\mathbf{R}$ from which the Euler angles α,β,γ have to be extracted to

[3] $\mathbf{R}$ is given by $\mathbf{R}_{j_2}\mathbf{R}_{i_1}^t$ where the columns of $\mathbf{R}_{i_k}$ are given by the local frames of point i in image $k \in \{1,2\}$, e.g. if the local frames are given by the normal $\boldsymbol{n}$ and the directions of minimal $\boldsymbol{e}^1$ and maximal $\boldsymbol{e}^2$ curvatures $\mathbf{R}_{i_k} = (\boldsymbol{n}_{i_k}, \boldsymbol{e}^1_{i_k}, \boldsymbol{e}^2_{i_k})$.

make an entry in the Hough table at the position corresponding to α, β, γ:

$$\mathbf{R} = \begin{pmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{pmatrix} \Rightarrow (\alpha, \beta, \gamma). \tag{5}$$

Since we know that $\mathbf{R}$ is a rotation matrix it should be orthogonal,

$$\boldsymbol{R}^t \boldsymbol{R} = \boldsymbol{1} = \boldsymbol{R}\boldsymbol{R}^t. \tag{6}$$

By ensuring that $\mathbf{R}$ is calculated from orthonormal frames we expect that this condition is satisfied. Nevertheless we have no guarantee that this is really the case due to numerical errors. The quaternion representation may be advantageous in this case, since for a given quaternion it is easy to find the next quaternion which represents a rotation.

Comparing (5) with (1) we especially get the following relations,

$$r_{22} = \cos\beta, \tag{7}$$

$$\begin{aligned} r_{12} &= \sin\beta\sin\alpha, & (8) \\ r_{02} &= \sin\beta\cos\alpha, & (9) \end{aligned}$$

$$\begin{aligned} r_{21} &= \sin\beta\sin\gamma, & (10) \\ r_{20} &= -\sin\beta\cos\gamma, & (11) \end{aligned}$$

Before calculating α, β and γ from (7)–(11) we would like to mention that the functions arccos and arcsin are calculated by the computer in the following ranges,

$$\begin{array}{ccccccccc} \varphi_1 & = & \arccos x & \Longrightarrow & 0 & \leq & \varphi_1 & \leq & \pi \\ \varphi_2 & = & \arcsin x & \Longrightarrow & -\frac{\pi}{2} & \leq & \varphi_2 & \leq & \frac{\pi}{2}. \end{array} \tag{12}$$

From (7) we get for β

$$\beta = \arccos r_{22}. \tag{13}$$

Since, following to (4), β lies in the range between 0 and π, $\sin\beta$ is equal to or greater than zero,

$$\sin\beta \geq 0. \tag{14}$$

From (8) we get

$$\sin\alpha = \frac{r_{12}}{\sin\beta}. \tag{15}$$

Note that, if β goes to zero we will run into trouble with this equation.

Since $\sin\beta \geq 0$ the sign of $\sin\alpha$ and therefore the range of possible solutions for α completely depends on the sign of r_{12}: if r_{12} is less than zero, $\sin\alpha$ will be also less than zero. Since, due to (2), α lies in the range between 0 and 2π, from $\sin\alpha < 0$ it follows that α is in the range between π and 2π. On the other hand the computer will give us a result for α in the range $-\frac{\pi}{2}$ and 0 when $\sin\alpha < 0$,

$$r_{12} < 0 \overset{(14),(15)}{\Longrightarrow} \sin\alpha < 0 \begin{cases} \overset{(12)}{\Longrightarrow} & -\frac{\pi}{2} \leq \alpha < 0 \\ \overset{\alpha\in[0,2\pi[}{\Longrightarrow} & \pi < \alpha < 2\pi \end{cases} \tag{16}$$

In the same way we get in case of $r_{12} \geq 0$:

$$r_{12} \geq 0 \overset{(14),(15)}{\Longrightarrow} \sin\alpha \geq 0 \begin{cases} \overset{(12)}{\Longrightarrow} & 0 \leq \alpha \leq \frac{\pi}{2} \\ \overset{\alpha\in[0,2\pi[}{\Longrightarrow} & 0 \leq \alpha \leq \pi \end{cases} \tag{17}$$

Similar conclusions can be made by using the second relation for α Eq.(9),

$$\cos\alpha = \frac{r_{02}}{\sin\beta}. \tag{18}$$

Performing a case distinction for r_{02} we get,

$$r_{02} < 0 \overset{(14),(18)}{\Longrightarrow} \cos\alpha < 0 \begin{cases} \overset{(12)}{\Longrightarrow} & \frac{\pi}{2} < \alpha \leq \pi \\ \overset{\alpha\in[0,2\pi[}{\Longrightarrow} & \frac{\pi}{2} < \alpha < \frac{3\pi}{2} \end{cases} \tag{19}$$

$$r_{02} \geq 0 \overset{(14),(18)}{\Longrightarrow} \cos\alpha \geq 0 \begin{cases} \overset{(12)}{\Longrightarrow} & 0 \leq \alpha \leq \frac{\pi}{2} \\ \overset{\alpha\in[0,2\pi[}{\Longrightarrow} & 0 \leq \alpha \leq \frac{\pi}{2} \\ & \vee \; \frac{3\pi}{2} \leq \alpha < 2\pi \end{cases} \tag{20}$$

Combining the conditions from (16),(17) and (19),(20) more precise conclusions can be made. For example, from (16) and (19) we get,

$$\begin{aligned} r_{12} < 0 \;\wedge\; r_{02} < 0 \overset{\alpha\in[0,2\pi[}{\Longrightarrow} \pi < \alpha < 2\pi \;\wedge\; \frac{\pi}{2} < \alpha < \frac{3\pi}{2} \\ \hookrightarrow \pi < \alpha < \frac{3\pi}{2}. \end{aligned} \tag{21}$$

On the other hand, as mentioned in (16), if α is calculated from (15) as

$$\alpha = \arcsin\frac{r_{12}}{\sin\beta} \tag{22}$$

the computer's result will be in the range between $-\frac{\pi}{2}$ and 0. To map this interval on the right interval from (21) we have to perform the following transformation,

$$\alpha \quad \rightarrow \quad \pi - \alpha. \tag{23}$$

This does not change the validity of (22) since, due to the addition theorem for sine,

$$\sin(\pi - \alpha) = \sin\alpha. \tag{24}$$

To summarize we have,

$$r_{12} < 0 \quad \wedge \quad r_{02} < 0 \quad : \qquad \alpha = \pi - \arcsin\frac{r_{12}}{\sin\beta}. \tag{25}$$

In the same way combining (16) and (20) we get

$$r_{12} < 0 \quad \wedge \quad r_{02} \geq 0 \quad \stackrel{\alpha\in[0,2\pi[}{\Longrightarrow} \qquad \pi < \alpha < 2\pi \qquad \wedge \quad \left(0 \leq \alpha \leq \frac{\pi}{2} \vee \frac{3\pi}{2} \leq \alpha < 2\pi\right)$$
$$\hookrightarrow \frac{3\pi}{2} \leq \alpha < 2\pi. \tag{26}$$

To map the computer's result of (22) in the range $[-\frac{\pi}{2}, 0[$ to the right interval from (26) we have to perform the following transformation

$$\alpha \quad \rightarrow \quad \alpha + 2\pi. \tag{27}$$

This does not change the validity of (22) since

$$\sin(\alpha + 2\pi) = \sin\alpha. \tag{28}$$

To summarize we have,

$$r_{12} < 0 \quad \wedge \quad r_{02} \geq 0 \quad : \qquad \alpha = \arcsin\frac{r_{12}}{\sin\beta} + 2\pi. \tag{29}$$

Combining (17) and (19) we get

$$r_{12} \geq 0 \quad \wedge \quad r_{02} < 0 \quad \stackrel{\alpha\in[0,2\pi[}{\Longrightarrow} \qquad 0 \leq \alpha \leq \pi \qquad \wedge \quad \frac{\pi}{2} < \alpha < \frac{3\pi}{2}$$
$$\hookrightarrow \frac{\pi}{2} < \alpha \leq \pi. \tag{30}$$

As mentioned in (17), if α is calculated from (22) the computer's result will be in the range between 0 and $\frac{\pi}{2}$ if $r_{12} \geq 0$. To map this interval on the right interval from (30) we have to perform the same transformation as in (23). Therefore

$$r_{12} \geq 0 \quad \wedge \quad r_{02} < 0 \quad : \qquad \alpha = \pi - \arcsin\frac{r_{12}}{\sin\beta}. \tag{31}$$

Finally, combining (17) and (20) we get

$$r_{12} \geq 0 \quad \wedge \quad r_{02} \geq 0 \quad \overset{\alpha\in[0,2\pi[}{\Longrightarrow} \quad 0 \leq \alpha \leq \pi \quad \wedge \quad \left(0 \leq \alpha \leq \tfrac{\pi}{2} \vee \tfrac{3\pi}{2} \leq \alpha < 2\pi\right)$$
$$\hookrightarrow 0 \leq \alpha \leq \tfrac{\pi}{2}. \tag{32}$$

Since the computer's result of α calculated from (22) is already in the same range (compare (17)), we do not have to transform α in this case. Therefore

$$r_{12} \geq 0 \quad \wedge \quad r_{02} \geq 0 \quad : \qquad \alpha = \arcsin \tfrac{r_{12}}{\sin\beta}. \tag{33}$$

To formulate an efficient algorithm for the calculation of α note that (25) and (31) can be combined to

$$r_{02} < 0 \quad : \qquad \alpha = \pi - \arcsin \tfrac{r_{12}}{\sin\beta}. \tag{34}$$

The essence of (29), (33) and (34) can be combined in the following pseudo-code:

> $\alpha = \arcsin\left(\frac{r_{12}}{\sin\beta}\right)$;
> IF $(r_{02} < 0)$ THEN $\alpha \leftarrow \pi - \alpha$;
> ELSE IF $(r_{12} < 0)$ THEN $\alpha \leftarrow \alpha + 2\pi$;

The same derivation as for α can be made for γ. This derivation can be shortened by comparing the relations for γ (10), (11) with the relations for α (8), (9): Substituting in (8) and (9) α by γ, r_{02} by $-r_{20}$ and r_{12} by r_{21} we get the relations (10), (11) that are the starting point for a derivation of the determination of γ. Therefore we can make the same substitutions in the pseudo-code for the determination of α to get an algorithm for the determination of γ:

> $\gamma = \arcsin\left(\frac{r_{21}}{\sin\beta}\right)$;
> IF $(-r_{20} < 0)$ THEN $\gamma \leftarrow \pi - \gamma$;
> ELSE IF $(r_{21} < 0)$ THEN $\gamma \leftarrow \gamma + 2\pi$;

β is calculated from (13). Since $\sin\beta$ is used for the determination of α and for the determination of γ it is useful to memorize this value in a separate variable b. In addition we would like to write $r_{20} > 0$ instead of $-r_{20} < 0$. In this way the complete algorithm for the determination of α, β and γ can be written as:

$\beta = \arccos r_{22};$
$b = \sin\beta;$

$\alpha = \arcsin\left(\frac{r_{12}}{b}\right);$
IF$(r_{02} < 0)$ THEN $\alpha \leftarrow \pi - \alpha;$
ELSE IF $(r_{12} < 0)$ THEN $\alpha \leftarrow \alpha + 2\pi;$

$\gamma = \arcsin\left(\frac{r_{21}}{b}\right);$
If $(r_{20} > 0)$ Then $\gamma \leftarrow \pi - \gamma;$
Else If $(r_{21} < 0)$ Then $\gamma \leftarrow \gamma + 2\pi;$

Although this algorithm seems to be very efficient we can reduce the number of operations if we calculate α and γ not from (8) respectively from (10) with the help of the arcsin function but from (9) respectively from (11) with the help of the arccos function. Let us derive the relations for α. From (9) (or (18)) we get

$$\alpha = \arccos\frac{r_{02}}{\sin\beta}. \tag{35}$$

As mentioned in (19) and (20) the computer's result for α will be in the range

$$\begin{array}{ccccccc} 0 & \leq & \alpha & \leq & \frac{\pi}{2} & \text{if} & r_{02} \geq 0, \\ \frac{\pi}{2} & < & \alpha & \leq & \pi & \text{if} & r_{02} < 0. \end{array} \tag{36}$$

On the other hand we know from (21), (26), (30) and (32)

$$\begin{array}{ccccccccc} r_{12} < 0 & \wedge & r_{02} < 0 & \Longrightarrow & \pi & < & \alpha & < & \frac{3\pi}{2} \\ r_{12} < 0 & \wedge & r_{02} \geq 0 & \Longrightarrow & \frac{3\pi}{2} & \leq & \alpha & < & 2\pi \\ r_{12} \geq 0 & \wedge & r_{02} < 0 & \Longrightarrow & \frac{\pi}{2} & < & \alpha & \leq & \pi \\ r_{12} \geq 0 & \wedge & r_{02} \geq 0 & \Longrightarrow & 0 & \leq & \alpha & \leq & \frac{\pi}{2}. \end{array} \tag{37}$$

To map the computer's result for α from (35) to the range $[0, 2\pi[$, α has to be defined as

$$\begin{array}{cccccc} r_{12} < 0 & \wedge & r_{02} < 0 & \Longrightarrow & \alpha & = & 2\pi - \arccos\frac{r_{02}}{\sin\beta} \\ r_{12} < 0 & \wedge & r_{02} \geq 0 & \Longrightarrow & \alpha & = & 2\pi - \arccos\frac{r_{02}}{\sin\beta} \\ r_{12} \geq 0 & \wedge & r_{02} < 0 & \Longrightarrow & \alpha & = & \arccos\frac{r_{02}}{\sin\beta} \\ r_{12} \geq 0 & \wedge & r_{02} \geq 0 & \Longrightarrow & \alpha & = & \arccos\frac{r_{02}}{\sin\beta}. \end{array} \tag{38}$$

This does not change the validity of (18) since, due to the addition theorem for cosine,

$$\cos(2\pi - \alpha) = \cos\alpha. \tag{39}$$

The essence of (38) can be combined in the following pseudo-code:

$\alpha = \arccos\left(\frac{r_{02}}{\sin\beta}\right)$;
IF $(r_{12} < 0)$ THEN $\alpha \leftarrow 2\pi - \alpha$;

To get the algorithm for the determination of γ we use the same procedure as described before: since the basic relations for γ (10), (11) can be received from the basic relations for α (8), (9) by substituting α by γ, r_{02} by $-r_{20}$ and r_{12} by r_{21} we only have to make the same substitutions in the pseudo-code for the determination of α:

$\gamma = \arccos\left(-\frac{r_{20}}{\sin\beta}\right)$;
IF $(r_{21} < 0)$ THEN $\gamma \leftarrow 2\pi - \gamma$;

Therefore the complete algorithm for the determination of α, β and γ can be written as:

$\beta = \arccos r_{22}$;
$b = \sin\beta$;

$\alpha = \arccos\left(\frac{r_{02}}{b}\right)$;
IF $(r_{12} < 0)$ THEN $\alpha \leftarrow 2\pi - \alpha$;

$\gamma = \arccos\left(-\frac{r_{20}}{b}\right)$;
IF $(r_{21} < 0)$ THEN $\gamma \leftarrow 2\pi - \gamma$;

2.3 Problems with Euler angle determination

As already mentioned before there is a problem with the described algorithm if β goes to zero. Since for the calculation of α as well as for the calculation of γ a term is divided by $b = \sin\beta$, the algorithm is not valid for $\beta = 0$ (or $\beta = \pi$) and will become unstable for $\beta \to 0$ (or $\beta \to \pi$). For $\beta \to 0$ ($\beta \to \pi$) also the other entries of the rotation matrix that are not used for the calculation of α, β and γ so far are not very helpful for the calculation of α and γ. To illustrate this fact we write the rotation matrix (1) for $\beta = 0$ ($\beta = \pi$),

$$\begin{aligned}
\mathbf{R}(\alpha, 0(\pi), \gamma) &= \begin{pmatrix} \overset{+}{\scriptstyle(-)} \cos\alpha\cos\gamma - \sin\alpha\sin\gamma & \overset{-}{\scriptstyle(+)} \cos\alpha\sin\gamma - \sin\alpha\cos\gamma & 0 \\ \overset{+}{\scriptstyle(-)} \sin\alpha\cos\gamma + \cos\alpha\sin\gamma & \overset{-}{\scriptstyle(+)} \sin\alpha\sin\gamma + \cos\alpha\cos\gamma & 0 \\ 0 & 0 & \overset{+}{\scriptstyle(-)} 1 \end{pmatrix} \\
&= \begin{pmatrix} \overset{+}{\scriptstyle(-)} \cos\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) & -\sin\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) & 0 \\ \overset{+}{\scriptstyle(-)} \sin\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) & \cos\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) & 0 \\ 0 & 0 & \overset{+}{\scriptstyle(-)} 1 \end{pmatrix}. \qquad (40)
\end{aligned}$$

Therefore for $\beta \to 0$ ($\beta \to \pi$) it is only possible to determine the sum (difference) of α and γ. α or γ itself are completely undetermined. This phenomena is known as the gimbal lock [3]. Another way to express this result is that all rotations $\mathbf{R}(\alpha, \Delta_\beta, \gamma)$ with $\alpha + \gamma = \text{const}$ respectively all rotations $\mathbf{R}(\alpha, \pi - \Delta_\beta, \gamma)$ with $\alpha - \gamma = \text{const}$ are neighbored. Thus very small variations in the entries of the given rotation matrix (5) will result in completely other parameters α and γ. Therefore it is in principle not possible to determine the Euler parameters with increasing accuracy if $\beta \to 0$ or $\beta \to \pi$ as it is needed in the hierarchical Hough table approach of [2].

2.4 Neighborhood structure

We would like to summarize the neighborhood structure of Euler angles: First, there are the trivial neighborhoods that do not make any problems:

- (α, β, γ) and $(\alpha \pm \Delta_\alpha, \beta, \gamma) \quad \forall \quad \alpha, \beta, \gamma$,
- (α, β, γ) and $(\alpha, \beta \pm \Delta_\beta, \gamma) \quad \forall \quad \alpha, \beta, \gamma$,
- (α, β, γ) and $(\alpha, \beta, \gamma \pm \Delta_\gamma) \quad \forall \quad \alpha, \beta, \gamma$.

Since, due to (2) and (3), α and γ are in the range from 0 to 2π and since $\sin\varphi = \sin(\varphi + 2\pi)$ and $\cos\varphi = \cos(\varphi + 2\pi)$ we also have the neighborhoods:

- $(\Delta_\alpha, \beta, \gamma)$ and $(2\pi - \Delta_\alpha, \beta, \gamma) \quad \forall \quad \beta, \gamma$,
- $(\alpha, \beta, \Delta_\gamma)$ and $(\alpha, \beta, 2\pi - \Delta_\gamma) \quad \forall \quad \alpha, \beta$.

More subtle are the neighborhoods mentioned in Subsec. 2.3,

- $(\alpha, \Delta_\beta, \gamma) \quad \forall \quad \alpha, \gamma : \quad \alpha + \gamma = \text{const}$,
- $(\alpha, \pi - \Delta_\beta, \gamma) \quad \forall \quad \alpha, \gamma : \quad \alpha - \gamma = \text{const}$.

As already mentioned in Subsec. 2.3 it is the Euler angle neighborhood structure for $\beta \to 0$ and $\beta \to \pi$ that results in problems with the Hough table approach of [2]. In the next subsection we present a solution to overcome these problems.

2.5 Solutions to neighborhood problems

The basic idea to avoid problems with the determination of the Euler angles α, β and γ from a given rotation matrix is to ensure that the rotation matrix does not correspond to an angle $\beta \to 0$ or $\beta \to \pi$. For a given rotation matrix it can be quickly analyzed from the entry $r_{22} = \cos\beta$ whether $\beta \to 0$ or $\beta \to \pi$: if $\|r_{22}\| \approx 1$ we will run into problems. Therefore in this case we transform the given rotation matrix by a constant rotation in such a way that there are no problems to extract the Euler angle parameters from the combined matrix. In this way we get unique parameters for the combined matrix. However, even with these unique parameters for the combined matrix it is not possible to get unique parameters for the original matrix. But fortunately in the Hough table approach of [2] we do not need the Euler parameters of the original matrix. To clarify this fact we shortly repeat the algorithm of the Hough table approach. The basic steps are:

1. calculate a rotation matrix (and a translation vector) from every possible point correspondences of the images to be registered,

2. calculate three (unique) parameters (p_1, p_2, p_3) from each rotation matrix,

3. increment a counter in the Hough table at position[4] $(t_x, t_y, t_z, p_1, p_2, p_3)$,

4. detect the parameter set that has most entries in the Hough table,

5. return the translation vector and the rotation matrix that correspond to the most frequently occuring parameter set.

Thus, although we need a unique parameter set for the rotation in between, we do not need any parameters for the rotation at the end. Therefore, if we transform each rotation matrix $\boldsymbol{R}$ from step 1 with a rotation matrix $\mathbf{R}_{\text{const}}$ we can get back the original rotation matrix $\boldsymbol{R}$ in step 5 just by multiplying the combined matrix $\boldsymbol{R}_{\text{const}}\boldsymbol{R}$ by $\boldsymbol{R}_{\text{const}}^{-1} = \boldsymbol{R}_{\text{const}}^{t}$ from the left.

As mentioned above it is quite easy to analyze whether a given rotation matrix will run into trouble with the $\beta \to 0$ or $\beta \to \pi$ problem. However, it is the principle of the Hough table approach that the same parameter set is used for all rotations and translations. Therefore it is not possible to detect a problematic matrix, transform it with a constant rotation $\boldsymbol{R}_{\text{const}}$, calculate the rotation parameters and increment a counter in the Hough table at the position of the parameters. We have to decide beforehand whether we want to transform all given rotation

[4] t_x, t_y and t_z are the translation parameters.

matrices by a constant rotation or not. But since we do not know beforehand whether the searched transformation (the transformation that should correspond to the Hough peak) will fail on the $\beta \to 0, \pi$ problem, we have to do both: we have to transform all given rotation matrices by a constant rotation and build up a Hough table for them and we have to extract the rotation parameters from the original rotation matrix and build up a separate Hough table for these parameters.

We would like to mention that instead of transforming each of the n^2 rotation matrices in step 1 with $\boldsymbol{R}_{\text{const}}$ (n is the number of points (respectively regions, compare [2]) in each of the images to be registered) it is also possible just to transform the n points (respectively $\approx n^2$ regions!) and local frames of the second image[5]. The rotation and translation from point $\boldsymbol{p}_{i_1}$ of the first image to point $\boldsymbol{p}_{j_2}$ of the second image is given by

$$\begin{aligned} \boldsymbol{R}_{i_1 \to j_2} &= \boldsymbol{R}_{j_2} \boldsymbol{R}_{i_1}^t, & (41) \\ \boldsymbol{t}_{i_1 \to j_2} &= \boldsymbol{p}_{j_2} - \boldsymbol{R}_{j_2} \boldsymbol{R}_{i_1}^t \boldsymbol{p}_{i_1} & (42) \end{aligned}$$

where the columns of $\boldsymbol{R}_{j_k}$ are given by the local frame of point j in image k. Therefore transforming the points and local frames of the second image by $\boldsymbol{R}_{\text{const}}$ results in the new rotations and translations

$$\begin{aligned} \boldsymbol{R}_{\text{const}} \boldsymbol{R}_{j_2} \boldsymbol{R}_{i_1}^t &= \boldsymbol{R}_{\text{const}} \boldsymbol{R}_{i_1 \to j_2} \quad , & (43) \\ \boldsymbol{R}_{\text{const}} \boldsymbol{p}_{j_2} - \boldsymbol{R}_{\text{const}} \boldsymbol{R}_{j_2} \boldsymbol{R}_{i_1}^t \boldsymbol{p}_{i_1} &= \boldsymbol{R}_{\text{const}} \boldsymbol{t}_{i_1 \to j_2} \quad . & (44) \end{aligned}$$

In this way we can register the parameters of the rotation matrix $\boldsymbol{R}_{\text{const}} \boldsymbol{R}_{i_1 \to j_2}$ and of the translation vector $\boldsymbol{R}_{\text{const}} \boldsymbol{t}_{i_1 \to j_2}$ in the Hough table, detect the parameter set with maximal entry in the Hough table, rebuild the rotation matrix (and the translation vector) from this parameter set and finally return the retransformed rotation matrix $\boldsymbol{R}_{i_1 \to j_2}$ and translation vector $\boldsymbol{t}_{i_1 \to j_2}$ by multiplying $\boldsymbol{R}_{\text{const}}^t$ from the left,

$$\begin{aligned} \boldsymbol{R}_{i_1 \to j_2} &= \boldsymbol{R}_{\text{const}}^t \left(\boldsymbol{R}_{\text{const}} \boldsymbol{R}_{i_1 \to j_2} \right) \quad , & (45) \\ \boldsymbol{t}_{1i \to j_2} &= \boldsymbol{R}_{\text{const}}^t \left(\boldsymbol{R}_{\text{const}} \boldsymbol{t}_{i_1 \to j_2} \right) \quad . & (46) \end{aligned}$$

Of course, separate Hough tables have to be build for the transformations following from the transformed points and frames and the transformations following from the original points and frames.

Till now we have shown that the problems resulting from the neighborhood structure for $\beta \to 0, \pi$ can be solved by an additional constant rotation $\boldsymbol{R}_{\text{const}}$. Now

[5] If the transformation from image 1 to image 2 is calculated the points and local frames of the second image have to be transformed (compare (43) and (44)).

we would like to give an example for a good choice of such a constant rotation matrix. Therefore we explicitly write down the matrix product $\boldsymbol{R}_{\text{const}}\boldsymbol{R}_{i_1\to j_2}$ for the case $\beta\to 0,\pi$ and compare the result with the rotation matrix in the Euler angle parameterization (1),

$$\boldsymbol{R}_{\text{const}}\boldsymbol{R}_{i_1\to j_2} = \begin{pmatrix} \tilde{r}_{00} & \tilde{r}_{01} & \tilde{r}_{02} \\ \tilde{r}_{10} & \tilde{r}_{11} & \tilde{r}_{12} \\ \tilde{r}_{20} & \tilde{r}_{21} & \tilde{r}_{22} \end{pmatrix} \begin{pmatrix} \cos\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) & -\sin\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) & 0 \\ (-)\sin\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) & \cos\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) & 0 \\ 0 & 0 & (-)1 \end{pmatrix} \tag{47}$$

$$= \begin{pmatrix} \tilde{r}_{00}\cos\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) \overset{+}{\scriptstyle(-)} \sin\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) & -\tilde{r}_{00}\sin\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) + \tilde{r}_{01}\cos\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) & (-)\tilde{r}_{02} \\ \tilde{r}_{10}\cos\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) \overset{+}{\scriptstyle(-)} \tilde{r}_{11}\sin\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) & -\tilde{r}_{10}\sin\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) + \tilde{r}_{11}\cos\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) & (-)\tilde{r}_{12} \\ \tilde{r}_{20}\cos\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) \overset{+}{\scriptstyle(-)} \tilde{r}_{21}\sin\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) & -\tilde{r}_{20}\sin\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) + \tilde{r}_{21}\cos\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) & (-)\tilde{r}_{22} \end{pmatrix} \tag{48}$$

$$\overset{!}{=} \begin{pmatrix} \cos\tilde{\alpha}\cos\tilde{\beta}\cos\tilde{\gamma} - \sin\tilde{\alpha}\sin\tilde{\gamma} & -\cos\tilde{\alpha}\cos\tilde{\beta}\sin\tilde{\gamma} - \sin\tilde{\alpha}\cos\tilde{\gamma} & \cos\tilde{\alpha}\sin\tilde{\beta} \\ \sin\tilde{\alpha}\cos\tilde{\beta}\cos\tilde{\gamma} + \cos\tilde{\alpha}\sin\tilde{\gamma} & -\sin\tilde{\alpha}\cos\tilde{\beta}\sin\tilde{\gamma} + \cos\tilde{\alpha}\cos\tilde{\gamma} & \sin\tilde{\alpha}\sin\tilde{\beta} \\ -\sin\tilde{\beta}\cos\tilde{\gamma} & \sin\tilde{\beta}\sin\tilde{\gamma} & \cos\tilde{\beta} \end{pmatrix} \tag{49}$$

Since $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are calculated as described in Subsec. 2.2 from the matrix entries $r_{02}, r_{12}, r_{22}, r_{20}, r_{21}$ we especially get from a comparison of (48) with (49),

$$(-)\tilde{r}_{22} = \cos\tilde{\beta}, \tag{50}$$

$$(-)\tilde{r}_{02} = \cos\tilde{\alpha}\sin\tilde{\beta}, \tag{51}$$

$$(-)\tilde{r}_{12} = \sin\tilde{\alpha}\sin\tilde{\beta}, \tag{52}$$

$$\tilde{r}_{20}\cos\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) \overset{+}{\scriptstyle(-)} \tilde{r}_{21}\sin\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) = -\cos\tilde{\gamma}\sin\tilde{\beta}, \tag{53}$$

$$-\tilde{r}_{20}\sin\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) + \tilde{r}_{21}\cos\left(\alpha \overset{+}{\scriptstyle(-)} \gamma\right) = \sin\tilde{\gamma}\sin\tilde{\beta}. \tag{54}$$

To avoid the gimbal lock problem in the matrix (49) we would like to have a value for $\tilde{\beta}$ so that $\sin\tilde{\beta}$ and $\cos\tilde{\beta}$ have the same values, i.e.

$$\tilde{\beta} = \frac{\pi}{4} \quad \Longrightarrow \quad \sin\tilde{\beta} = \cos\tilde{\beta} = \frac{1}{\sqrt{2}}. \tag{55}$$

This can be reached by appropriately choosing $\tilde{r}_{ij}$. Inserting (55) in (50–54) results in

$$(-)\tilde{r}_{22} = \frac{1}{\sqrt{2}}, \tag{56}$$

$$
\begin{aligned}
{\scriptstyle (-)}\tilde{r}_{02} &= \cos\tilde{\alpha}\frac{1}{\sqrt{2}}, && (57)\\
{\scriptstyle (-)}\tilde{r}_{12} &= \sin\tilde{\alpha}\frac{1}{\sqrt{2}}, && (58)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{r}_{20}\cos\left(\alpha \genfrac{}{}{0pt}{}{+}{(-)} \gamma\right) \genfrac{}{}{0pt}{}{+}{(-)} \tilde{r}_{21}\sin\left(\alpha \genfrac{}{}{0pt}{}{+}{(-)} \gamma\right) &= -\cos\tilde{\gamma}\frac{1}{\sqrt{2}}, && (59)\\
-\tilde{r}_{20}\sin\left(\alpha \genfrac{}{}{0pt}{}{+}{(-)} \gamma\right) + \tilde{r}_{21}\cos\left(\alpha \genfrac{}{}{0pt}{}{+}{(-)} \gamma\right) &= \sin\tilde{\gamma}\frac{1}{\sqrt{2}}. && (60)
\end{aligned}
$$

Till now only $\tilde{r}_{22}$ is determined through (56). Thus there is additional freedom in choosing $\boldsymbol{R}_{\text{const}}$. We would like to choose $\tilde{\alpha}$ in (57) and (58) again in a way so that $\sin\tilde{\alpha}$ and $\cos\tilde{\alpha}$ have the same values, i.e.

$$
\tilde{\alpha} = \frac{\pi}{4} \quad \Longrightarrow \quad \sin\tilde{\alpha} = \cos\tilde{\alpha} = \frac{1}{\sqrt{2}}. \tag{61}
$$

Inserting (61) in (57–58) we get

$$
\begin{aligned}
{\scriptstyle (-)}\tilde{r}_{02} &= \frac{1}{2}, && (62)\\
{\scriptstyle (-)}\tilde{r}_{12} &= \frac{1}{2}, && (63)
\end{aligned}
$$

It is not possible to choose $\tilde{\gamma}$ in (59) and (60) with the same freedom as $\tilde{\alpha}$ and $\tilde{\beta}$ since $\tilde{\gamma}$ depends on $\alpha \genfrac{}{}{0pt}{}{+}{(-)} \gamma$ (and not only on $\tilde{r}_{20}$ and $\tilde{r}_{21}$). Nevertheless there is some freedom in choosing $\tilde{r}_{20}$ and $\tilde{r}_{21}$. There is only one constraint we have to observe: from the orthogonality of $\boldsymbol{R}_{\text{const}}$, i.e. $\boldsymbol{R}_{\text{const}}\boldsymbol{R}^t_{\text{const}} = \boldsymbol{1}$, we get 9 equations that all reduce to the same equation when we insert the already chosen values for $\tilde{r}_{02}, \tilde{r}_{21}$ and $\tilde{r}_{22}$ from (62), (63) and (56) in $\boldsymbol{R}_{\text{const}}$,

$$
\tilde{r}^2_{20} + \tilde{r}^2_{21} = \frac{1}{2}. \tag{64}
$$

This equation follows directly from the component $\left(\boldsymbol{R}_{\text{const}}\boldsymbol{R}^t_{\text{const}}\right)_{22}$. To prove that the equations following from the 8 other components of $\boldsymbol{R}_{\text{const}}\boldsymbol{R}^t_{\text{const}}$ result in the same equation (64) in a first step all components of $\boldsymbol{R}_{\text{const}}$ have to be expressed through $\tilde{r}_{20}, \tilde{r}_{21}, \tilde{r}_{22}, \tilde{r}_{02}$ and $\tilde{r}_{12}$. This can be easily done by using (1) and (5),

$$
\begin{pmatrix} \tilde{r}_{00} & \tilde{r}_{01} & \tilde{r}_{02} \\ \tilde{r}_{10} & \tilde{r}_{11} & \tilde{r}_{12} \\ \tilde{r}_{20} & \tilde{r}_{21} & \tilde{r}_{22} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} \cos\bar{\alpha}\cos\bar{\beta}\cos\bar{\gamma} - \sin\bar{\alpha}\sin\bar{\gamma} & -\cos\bar{\alpha}\cos\bar{\beta}\sin\bar{\gamma} - \sin\bar{\alpha}\cos\bar{\gamma} & \cos\bar{\alpha}\sin\bar{\beta} \\ \sin\bar{\alpha}\cos\bar{\beta}\cos\bar{\gamma} + \cos\bar{\alpha}\sin\bar{\gamma} & -\sin\bar{\alpha}\cos\bar{\beta}\sin\bar{\gamma} + \cos\bar{\alpha}\cos\bar{\gamma} & \sin\bar{\alpha}\sin\bar{\beta} \\ -\sin\bar{\beta}\cos\bar{\gamma} & \sin\bar{\beta}\sin\bar{\gamma} & \cos\bar{\beta} \end{pmatrix} \tag{65}
$$

By inserting the relations following from (65)

$$\cos\bar{\alpha}\sin\bar{\beta} = \tilde{r}_{02} \qquad \Longrightarrow \qquad \cos\bar{\alpha} = \frac{r_{02}}{\sin\beta}, \tag{66}$$

$$\sin\bar{\alpha}\sin\bar{\beta} = \tilde{r}_{12} \qquad \Longrightarrow \qquad \sin\bar{\alpha} = \frac{r_{12}}{\sin\beta}, \tag{67}$$

$$-\cos\bar{\gamma}\sin\bar{\beta} = \tilde{r}_{20} \qquad \Longrightarrow \qquad \cos\bar{\gamma} = -\frac{r_{20}}{\sin\beta}, \tag{68}$$

$$\sin\bar{\gamma}\sin\bar{\beta} = \tilde{r}_{21} \qquad \Longrightarrow \qquad \sin\bar{\gamma} = \frac{r_{21}}{\sin\beta}, \tag{69}$$

$$\cos\bar{\beta} = \tilde{r}_{22} \tag{70}$$

$$\overset{\sin^2\bar{\beta}+\cos^2\bar{\beta}=1}{\hookrightarrow} \qquad \sin^2\bar{\beta} = 1-\tilde{r}_{22}^2, \tag{71}$$

in $(\boldsymbol{R}_{\text{const}})_{00}$, $(\boldsymbol{R}_{\text{const}})_{01}$, $(\boldsymbol{R}_{\text{const}})_{10}$ and $(\boldsymbol{R}_{\text{const}})_{11}$ from (65) we get

$$\boldsymbol{R}_{\text{const}} = \begin{pmatrix} \frac{-\tilde{r}_{02}\tilde{r}_{20}\tilde{r}_{22}-\tilde{r}_{12}\tilde{r}_{21}}{1-\tilde{r}_{22}^2} & \frac{\tilde{r}_{12}\tilde{r}_{20}-\tilde{r}_{02}\tilde{r}_{22}\tilde{r}_{21}}{1-\tilde{r}_{22}^2} & \tilde{r}_{02} \\ \frac{\tilde{r}_{21}\tilde{r}_{02}-\tilde{r}_{20}\tilde{r}_{22}\tilde{r}_{12}}{1-\tilde{r}_{22}^2} & \frac{-\tilde{r}_{12}\tilde{r}_{21}\tilde{r}_{22}-\tilde{r}_{02}\tilde{r}_{20}}{1-\tilde{r}_{22}^2} & \tilde{r}_{12} \\ \tilde{r}_{20} & \tilde{r}_{21} & \tilde{r}_{22} \end{pmatrix} \tag{72}$$

In this way it is straightforward to see that the nine equations following from

$$
\begin{aligned}
\mathbf{1} &\overset{!}{=} \boldsymbol{R}_{\text{const}} \boldsymbol{R}^{t}_{\text{const}} \\
&= \begin{pmatrix} \frac{-\tilde r_{02}\tilde r_{20}\tilde r_{22}-\tilde r_{12}\tilde r_{21}}{1-\tilde r_{22}^2} & \frac{\tilde r_{12}\tilde r_{20}-\tilde r_{02}\tilde r_{22}\tilde r_{21}}{1-\tilde r_{22}^2} & \tilde r_{02} \\ \frac{\tilde r_{21}\tilde r_{02}-\tilde r_{20}\tilde r_{22}\tilde r_{12}}{1-\tilde r_{22}^2} & \frac{-\tilde r_{12}\tilde r_{21}\tilde r_{22}-\tilde r_{02}\tilde r_{20}}{1-\tilde r_{22}^2} & \tilde r_{12} \\ \tilde r_{20} & \tilde r_{21} & \tilde r_{22} \end{pmatrix} \begin{pmatrix} \frac{-\tilde r_{02}\tilde r_{20}\tilde r_{22}-\tilde r_{12}\tilde r_{21}}{1-\tilde r_{22}^2} & \frac{\tilde r_{21}\tilde r_{02}-\tilde r_{20}\tilde r_{22}\tilde r_{12}}{1-\tilde r_{22}^2} & \tilde r_{20} \\ \frac{\tilde r_{12}\tilde r_{20}-\tilde r_{02}\tilde r_{22}\tilde r_{21}}{1-\tilde r_{22}^2} & \frac{-\tilde r_{12}\tilde r_{21}\tilde r_{22}-\tilde r_{02}\tilde r_{20}}{1-\tilde r_{22}^2} & \tilde r_{21} \\ \tilde r_{02} & \tilde r_{12} & \tilde r_{22} \end{pmatrix} \\
&= \begin{pmatrix}
\left(\frac{-\tilde r_{02}\tilde r_{20}\tilde r_{22}-\tilde r_{12}\tilde r_{21}}{1-\tilde r_{22}^2}\right)^2 + \left(\frac{\tilde r_{12}\tilde r_{20}-\tilde r_{02}\tilde r_{22}\tilde r_{21}}{1-\tilde r_{22}^2}\right)^2 + \tilde r_{02}^2 & \frac{(-\tilde r_{02}\tilde r_{20}\tilde r_{22}-\tilde r_{12}\tilde r_{21})(\tilde r_{21}\tilde r_{02}-\tilde r_{20}\tilde r_{22}\tilde r_{12})}{(1-\tilde r_{22}^2)^2} + \frac{(\tilde r_{12}\tilde r_{20}-\tilde r_{02}\tilde r_{22}\tilde r_{21})(-\tilde r_{12}\tilde r_{21}\tilde r_{22}-\tilde r_{02}\tilde r_{20})}{(1-\tilde r_{22}^2)^2} + \tilde r_{02}\tilde r_{12} & \frac{-\tilde r_{02}\tilde r_{20}\tilde r_{22}-\tilde r_{12}\tilde r_{21}}{1-\tilde r_{22}^2}\tilde r_{20} + \frac{\tilde r_{12}\tilde r_{20}-\tilde r_{02}\tilde r_{22}\tilde r_{21}}{1-\tilde r_{22}^2}\tilde r_{21} + \tilde r_{02}\tilde r_{22} \\
\frac{(-\tilde r_{02}\tilde r_{20}\tilde r_{22}-\tilde r_{12}\tilde r_{21})(\tilde r_{21}\tilde r_{02}-\tilde r_{20}\tilde r_{22}\tilde r_{12})}{(1-\tilde r_{22}^2)^2} + \frac{(\tilde r_{12}\tilde r_{20}-\tilde r_{02}\tilde r_{22}\tilde r_{21})(-\tilde r_{12}\tilde r_{21}\tilde r_{22}-\tilde r_{02}\tilde r_{20})}{(1-\tilde r_{22}^2)^2} + \tilde r_{02}\tilde r_{12} & \left(\frac{\tilde r_{21}\tilde r_{02}-\tilde r_{20}\tilde r_{22}\tilde r_{12}}{1-\tilde r_{22}^2}\right)^2 + \left(\frac{-\tilde r_{12}\tilde r_{21}\tilde r_{22}-\tilde r_{02}\tilde r_{20}}{1-\tilde r_{22}^2}\right)^2 + \tilde r_{21}^2 & \frac{\tilde r_{21}\tilde r_{02}-\tilde r_{20}\tilde r_{22}\tilde r_{12}}{1-\tilde r_{22}^2}\tilde r_{20} + \frac{-\tilde r_{12}\tilde r_{21}\tilde r_{22}-\tilde r_{02}\tilde r_{20}}{1-\tilde r_{22}^2}\tilde r_{21} + \tilde r_{12}\tilde r_{22} \\
\frac{-\tilde r_{02}\tilde r_{20}\tilde r_{22}-\tilde r_{12}\tilde r_{21}}{1-\tilde r_{22}^2}\tilde r_{20} + \frac{\tilde r_{12}\tilde r_{20}-\tilde r_{02}\tilde r_{22}\tilde r_{21}}{1-\tilde r_{22}^2}\tilde r_{21} + \tilde r_{02}\tilde r_{22} & \frac{\tilde r_{21}\tilde r_{02}-\tilde r_{20}\tilde r_{22}\tilde r_{12}}{1-\tilde r_{22}^2}\tilde r_{20} + \frac{-\tilde r_{12}\tilde r_{21}\tilde r_{22}-\tilde r_{02}\tilde r_{20}}{1-\tilde r_{22}^2}\tilde r_{21} + \tilde r_{12}\tilde r_{22} & \tilde r_{20}^2 + \tilde r_{21}^2 + \tilde r_{22}^2
\end{pmatrix}
\end{aligned} \tag{73}
$$

all reduce to (64) when we insert $\tilde r_{02}$ and $\tilde r_{12}$ from (62) and (63).

Now we come back to (64). This condition is satisfied e.g. by choosing

$$
\tilde r_{20} = \tilde r_{21} = \frac{1}{2}. \tag{74}
$$

Therefore we finally get from (56), (62), (63), (72) and (74) the reasonable choice for $\boldsymbol{R}_{\text{const}}$

$$
\boldsymbol{R}_{\text{const}} = \begin{pmatrix} -\frac{1+\sqrt{2}}{2\sqrt{2}} & \frac{\sqrt{2}-1}{2\sqrt{2}} & \frac{1}{2} \\ \frac{\sqrt{2}-1}{2\sqrt{2}} & -\frac{1+\sqrt{2}}{2\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{75}
$$

There is another possibility to avoid problems with the determination of the Euler angles if $\beta \to 0$ or $\beta \to \pi$. Since, as we will see later, other parameterizations run into problems at parameter configurations belonging to other rotations, we can calculate from each rotation matrix two parameter sets belonging to two different parameterizations and register these parameters in two different Hough tables. In this way we save the extra rotation with $\boldsymbol{R}_{\text{const}}$ from above and avoid the gimbal lock problem in at least one Hough table.

An even better way to avoid the problems with the gimbal lock is to choose a parameterization that does not suffer from this problem. As will be seen later the quaternion representation is such a parameterization. Since the quaternion representation has another neighboring problem that makes the construction of the Hough table in certain cases very difficult and since the quaternion representation leads to a memory consuming Hough table due to its topology, we propose to use the quaternion parameterization only in the first iteration of the Hough table approach of [2], detect the Hough peak and then alternate to an Euler angle like parameterization that has no gimbal lock problems in the neighborhood of the Hough peak. In this way no extra calculations are necessary to avoid problems with the gimbal lock.

3 Euler angle like neighborhood structure

The definition of Euler angles is not uniquely handled in the literature. In Subsec. 2.1 Eq. (1) we have defined the Euler angles following to [4] by

$$\boldsymbol{R}_{zyz}(\alpha,\beta,\gamma) = \boldsymbol{R}_z(\alpha)\,\boldsymbol{R}_y(\beta)\,\boldsymbol{R}_z(\gamma) \qquad 0 \le \alpha,\gamma < 2\pi, \quad 0 \le \beta \le \pi \tag{76}$$

where $\boldsymbol{R}_{\hat{n}}(\varphi)$ describes a rotation where the unit vector $\hat{n}$ specifies the direction of the axis of rotation (here y and z axis of the given coordinate system) and φ the angle of rotation around that axis. On the other hand in [1] the Euler angles are defined by

$$\boldsymbol{R}_{zxz}(\alpha,\beta,\gamma) = \boldsymbol{R}_z(\alpha)\,\boldsymbol{R}_x(\beta)\,\boldsymbol{R}_z(\gamma) \qquad 0 \le \alpha,\gamma < 2\pi, \quad 0 \le \beta \le \pi. \tag{77}$$

These parameterizations have the gimbal lock problem for $\beta \to 0$ and $\beta \to \pi$. Therefore in situations where we are only interested in small transformations, i.e. transformations near the identity with Euler parameters close to 0, these parameterizations are not a good choice. In such cases it may be advantageous to use the Euler like parameterizations

$$\boldsymbol{R}_{zyx}(\alpha,\beta,\gamma) = \boldsymbol{R}_x(\gamma)\,\boldsymbol{R}_y(\beta)\,\boldsymbol{R}_z(\alpha) \qquad 0 \le \alpha,\gamma < 2\pi, \quad -\frac{\pi}{2} \le \beta \le \frac{\pi}{2} \tag{78}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta & -\sin\alpha\cos\beta & \sin\beta \\ \cos\alpha\sin\beta\sin\gamma+\sin\alpha\cos\gamma & -\sin\alpha\sin\beta\sin\gamma+\cos\alpha\cos\gamma & -\cos\beta\sin\gamma \\ -\cos\alpha\sin\beta\cos\gamma+\sin\alpha\sin\gamma & \sin\alpha\sin\beta\cos\gamma+\cos\alpha\sin\gamma & \cos\beta\cos\gamma \end{pmatrix} \tag{79}$$

or

$$\boldsymbol{R}_{xyz}(\alpha,\beta,\gamma) = \boldsymbol{R}_z(\gamma)\boldsymbol{R}_y(\beta)\boldsymbol{R}_x(\alpha) \qquad 0 \le \alpha,\gamma < 2\pi, \quad -\frac{\pi}{2} \le \beta \le \frac{\pi}{2} \tag{80}$$

$$= \begin{pmatrix} \cos\beta\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \cos\beta\sin\gamma & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma \\ -\sin\beta & \sin\alpha\cos\beta & \cos\alpha\cos\beta \end{pmatrix} \tag{81}$$

We would like to derive algorithms for the extraction of the angle parameters from rotation matrices for these two parameterizations in the following subsections.

3.1 Extraction of angle parameters from rotation matrices: ZYX case

Using $\boldsymbol{R}_{zyx}(\alpha,\beta,\gamma)$ in (78) in the form (5) we especially get from (79)

$$r_{02} = \sin\beta, \tag{82}$$

$$r_{00} = \cos\alpha\cos\beta, \tag{83}$$

$$r_{01} = -\sin\alpha\cos\beta, \tag{84}$$

$$r_{12} = -\cos\beta\sin\gamma, \tag{85}$$

$$r_{22} = \cos\beta\cos\gamma, \tag{86}$$

Since, following to (78), β lies in the range between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, $\cos\beta$ is equal to or greater than 0,

$$\cos\beta \ge 0. \tag{87}$$

From (83) we get

$$\cos\alpha = \frac{r_{00}}{\cos\beta}. \tag{88}$$

Since $\cos\beta \ge 0$ the sign of $\cos\alpha$ and therefore the range of possible solutions for α completely depends on the sign of r_{00}: if r_{00} is less than zero, $\cos\alpha$ will be also less than zero. Since, due to (78), α lies in the range between 0 and 2π, from $\cos\alpha < 0$ it follows that α is in the range between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. On the other hand, from $\cos\alpha \geq 0$ it follows that α is either in the range between 0 and $\frac{\pi}{2}$ or in the range between $\frac{3\pi}{2}$ and 2π,

$$r_{00} < 0 \overset{(87),(88)}{\Longrightarrow} \cos\alpha < 0 \overset{\alpha\in[0,2\pi[}{\Longrightarrow} \frac{\pi}{2} < \alpha < \frac{3\pi}{2}, \tag{89}$$

$$r_{00} \geq 0 \overset{(87),(88)}{\Longrightarrow} \cos\alpha \geq 0 \overset{\alpha\in[0,2\pi[}{\Longrightarrow} 0 \leq \alpha \leq \frac{\pi}{2} \vee \frac{3\pi}{2} \leq \alpha < 2\pi. \tag{90}$$

Similar conclusions can be made by using the second relation for α Eq. (84),

$$\sin\alpha = -\frac{r_{01}}{\cos\beta}. \tag{91}$$

Performing a case distinction for r_{01} we get,

$$r_{01} \leq 0 \overset{(87),(91)}{\Longrightarrow} \sin\alpha \geq 0 \overset{\alpha\in[0,2\pi[}{\Longrightarrow} 0 \leq \alpha \leq \pi, \tag{92}$$

$$r_{01} > 0 \overset{(87),(91)}{\Longrightarrow} \sin\alpha < 0 \overset{\alpha\in[0,2\pi[}{\Longrightarrow} \pi < \alpha < 2\pi. \tag{93}$$

Combining the relations from (89), (90), (92) and (93) more precise conclusions can be made,

$$\begin{aligned}
r_{00} < 0 \ \wedge \ r_{01} \leq 0 &\overset{(89),(92)}{\Longrightarrow} \tfrac{\pi}{2} < \alpha < \tfrac{3\pi}{2} \ \wedge \ 0 \leq \alpha \leq \pi \\
&\hookrightarrow \tfrac{\pi}{2} < \alpha \leq \pi, \\
r_{00} < 0 \ \wedge \ r_{01} > 0 &\overset{(89),(93)}{\Longrightarrow} \tfrac{\pi}{2} < \alpha < \tfrac{3\pi}{2} \ \wedge \ \pi < \alpha < 2\pi \\
&\hookrightarrow \pi < \alpha < \tfrac{3\pi}{2}, \\
r_{00} \geq 0 \ \wedge \ r_{01} \leq 0 &\overset{(90),(92)}{\Longrightarrow} \left(0 \leq \alpha \leq \tfrac{\pi}{2} \ \vee \ \tfrac{3\pi}{2} \leq \alpha < 2\pi\right) \ \wedge \ 0 \leq \alpha \leq \pi \\
&\hookrightarrow 0 \leq \alpha \leq \tfrac{\pi}{2}, \\
r_{00} \geq 0 \ \wedge \ r_{01} \geq 0 &\overset{(90),(93)}{\Longrightarrow} \left(0 \leq \alpha \leq \tfrac{\pi}{2} \ \vee \ \tfrac{3\pi}{2} \leq \alpha < 2\pi\right) \ \wedge \ \pi < \alpha < 2\pi \\
&\hookrightarrow \tfrac{3\pi}{2} \leq \alpha < 2\pi,
\end{aligned} \tag{94}$$

When computing α from (88) respectively from

$$\alpha = \arccos \frac{r_{00}}{\cos \beta} \tag{95}$$

the computer's result will be in the range $[0, \pi]$ following to (12). To be precise α will be in the range

$$\begin{array}{ccccc} 0 & \leq & \alpha & \leq & \frac{\pi}{2} \qquad \text{if} \quad r_{00} \geq 0, \\ \frac{\pi}{2} & < & \alpha & \leq & \pi \qquad \text{if} \quad r_{00} < 0. \end{array} \tag{96}$$

To map the computer's result to the range $[0, 2\pi[$, α has to be defined as

$$\begin{array}{lllll} r_{00} < 0 & \wedge & r_{01} \leq 0 & \Longrightarrow & \alpha = \arccos \frac{r_{00}}{\cos \beta}, \\ r_{00} < 0 & \wedge & r_{01} > 0 & \Longrightarrow & \alpha = 2\pi - \arccos \frac{r_{00}}{\cos \beta}, \\ r_{00} \geq 0 & \wedge & r_{01} \leq 0 & \Longrightarrow & \alpha = \arccos \frac{r_{00}}{\cos \beta}, \\ r_{00} \geq 0 & \wedge & r_{01} > 0 & \Longrightarrow & \alpha = 2\pi - \arccos \frac{r_{00}}{\cos \beta}. \end{array} \tag{97}$$

Again, this does not change the validity of (88) due to (39).

The essence of (97) can be combined in the following pseudo-code:

$\alpha = \arccos \left(\frac{r_{00}}{\cos \beta} \right);$
IF $(r_{01} > 0)$ THEN $\alpha \leftarrow 2\pi - \alpha;$

To get the algorithm for the determination of γ we proceed as in Subsec. 2.2: since the basic relations for γ (85), (86) can be received from the basic relations for α (83), (84) by substituting α by γ, r_{00} by r_{22} and r_{01} by r_{12} we only have to make the same substitutions in the pseudo-code for the determination of α:

$\gamma = \arccos \left(\frac{r_{22}}{\cos \beta} \right);$
IF $(r_{12} > 0)$ THEN $\gamma \leftarrow 2\pi - \gamma;$

Therefore the complete algorithm for the determination of α, β and γ can be written as:

$\beta = \arcsin r_{02};$
$b = \cos \beta;$

$\alpha = \arccos \left(\frac{r_{00}}{b} \right);$
IF $(r_{01} > 0)$ THEN $\alpha \leftarrow 2\pi - \alpha;$

$\gamma = \arccos \left(\frac{r_{22}}{b} \right);$
IF $(r_{12} > 0)$ THEN $\gamma \leftarrow 2\pi - \gamma;$

3.2 Extraction of angle parameters from rotation matrices: XYZ case

Comparing $\boldsymbol{R}_{zyx}(\alpha,\beta,\gamma)$ in the ZYX case (79) with $\boldsymbol{R}_{xyz}(\alpha,\beta,\gamma)$ in the XYZ case (81) we recognize that the matrix entries, that are relevant for the parameter extraction in the ZYX case, $r_{00}, r_{01}, r_{02}, r_{12}$ and r_{22} are quite similar to the matrix entries $r_{00}, r_{10}, r_{20}, r_{21}$ and r_{22} in the XYZ case:

$$\begin{array}{lclcl} r_{00} & \text{in (79)} & = & r_{22} & \text{in (81)} \\ r_{01} & \text{in (79)} & = & -r_{21} & \text{in (81)} \\ r_{02} & \text{in (79)} & = & -r_{20} & \text{in (81)} \\ r_{12} & \text{in (79)} & = & -r_{10} & \text{in (81)} \\ r_{22} & \text{in (79)} & = & r_{00} & \text{in (81)} \end{array} \tag{98}$$

Therefore if we make the substitutions induced by (98) in the pseudo code for the determination of angle parameters in the ZYX case we get an algorithm for the determination of the angle parameters in the XYZ case. In addition we prefer to use $r_{21} < 0$ and $r_{01} < 0$ instead of $-r_{21} > 0$ and $-r_{10} > 0$:

$\beta = \arcsin(-r_{20})$;
$b = \cos\beta$;

$\alpha = \arccos\left(\frac{r_{22}}{b}\right)$;
IF $(r_{21} < 0)$ THEN $\alpha \leftarrow 2\pi - \alpha$;

$\gamma = \arccos\left(\frac{r_{00}}{b}\right)$;
IF $(r_{10} < 0)$ THEN $\gamma \leftarrow 2\pi - \gamma$;

3.3 Problems with angle parameter determination

Of course, the Euler angle like parameterizations also suffer from the gimbal lock. Although the gimbal lock problem does not occur at $\beta \to 0$ and $\beta \to \pi$ we run into problems for $\beta \to -\frac{\pi}{2}$ and $\beta \to \frac{\pi}{2}$ since in the calculation of α and γ a term is divided by $\cos\beta$ that is 0 for $\beta = -\frac{\pi}{2}$ and $\beta = \frac{\pi}{2}$. As for the usual Euler angles in Subsec. 2.3 for the critical values of β the other entries in the rotation matrix that are not used for the calculation of α, β and γ so far are not very helpful for the calculation of α and γ. For $\beta = -\frac{\pi}{2}$ ($\beta = \frac{\pi}{2}$) we have

$$\mathbf{R}_{zyx}(\alpha, \underset{(+)}{-}\frac{\pi}{2}, \gamma) = \begin{pmatrix} 0 & 0 & \underset{(+)}{-}1 \\ \underset{(+)}{-}\cos\alpha\sin\gamma + \sin\alpha\cos\gamma & \underset{(-)}{+}\sin\alpha\sin\gamma + \cos\alpha\cos\gamma & 0 \\ \underset{(-)}{+}\cos\alpha\cos\gamma + \sin\alpha\sin\gamma & \underset{(+)}{-}\sin\alpha\cos\gamma + \cos\alpha\sin\gamma & 0 \end{pmatrix}$$

$$
= \begin{pmatrix} 0 & 0 & \underset{(+)}{-}1 \\ \sin\left(\alpha \underset{(+)}{-} \gamma\right) & \cos\left(\alpha \underset{(+)}{-} \gamma\right) & 0 \\ \underset{(-)}{+}\cos\left(\alpha \underset{(+)}{-} \gamma\right) & \underset{(+)}{-}\sin\left(\alpha \underset{(+)}{-} \gamma\right) & 0 \end{pmatrix} \tag{99}
$$

and

$$
\begin{aligned}
\mathbf{R}_{xyz}(\alpha, \underset{(+)}{-}\frac{\pi}{2}, \gamma) &= \begin{pmatrix} 0 & \underset{(+)}{-}\sin\alpha\cos\gamma - \cos\alpha\sin\gamma & \underset{(+)}{-}\cos\alpha\cos\gamma + \sin\alpha\sin\gamma \\ 0 & \underset{(+)}{-}\sin\alpha\sin\gamma + \cos\alpha\cos\gamma & \underset{(+)}{-}\cos\alpha\sin\gamma - \sin\alpha\cos\gamma \\ \underset{(-)}{+}1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \underset{(+)}{-}\sin\left(\alpha \underset{(-)}{+} \gamma\right) & \underset{(+)}{-}\cos\left(\alpha \underset{(-)}{+} \gamma\right) \\ 0 & \cos\left(\alpha \underset{(-)}{+} \gamma\right) & -\sin\left(\alpha \underset{(-)}{+} \gamma\right) \\ \underset{(-)}{+}1 & 0 & 0 \end{pmatrix}. \qquad (100)
\end{aligned}
$$

Therefore for $\beta \to \pm\frac{\pi}{2}$ it is only possible to determine the sum respectively difference of α and γ. As already mentioned in the context of the usual Euler angles in Subsec. 2.3 another way to express this result is that all rotations $\boldsymbol{R}_{zyx}\left(\alpha, \underset{(+)}{-}\frac{\pi}{2} \underset{(-)}{+} \Delta_\beta, \gamma\right)$ with $\alpha \underset{(+)}{-} \gamma = \text{const}$ respectively all rotations $\boldsymbol{R}_{xyz}\left(\alpha, \underset{(+)}{-}\frac{\pi}{2} \underset{(-)}{+} \Delta_\beta, \gamma\right)$ with $\alpha \underset{(-)}{+} \gamma = \text{const}$ are neighbored. All other neighborhoods are the same as described for the usual Euler angles in Subsec. 2.4.

4 Axis and angle neighborhood structure

In the axis and angle parameterization of rotations a rotation is characterized by an axis of rotation $\hat{\boldsymbol{n}}$ and an angle of rotation ψ around that axis: $\boldsymbol{R}_{\hat{n}}(\psi)$ (see Fig. 1). The axis of rotation is determined by the polar and azimuthal angles (θ, φ) of its direction with

$$
0 \leq \quad \theta \quad \leq \pi, \tag{101}
$$

$$
0 \leq \quad \varphi \quad < 2\pi \quad . \tag{102}
$$

For the angle of rotation ψ we have

$$
0 \leq \psi \leq \pi. \tag{103}
$$

There is a redundancy in this parameterization:

$$
\boldsymbol{R}_{-\hat{n}}(\pi) = \boldsymbol{R}_{\hat{n}}(\pi) \quad . \tag{104}
$$

The structure of the parameter space can be visualized by associating each rotation with a three dimensional vector $\boldsymbol{v} = \psi\hat{\boldsymbol{n}}$ pointing in the direction $\hat{\boldsymbol{n}}$ with magnitude equal to ψ. The tips of these vectors fill a 3-dimensional sphere of radius π.

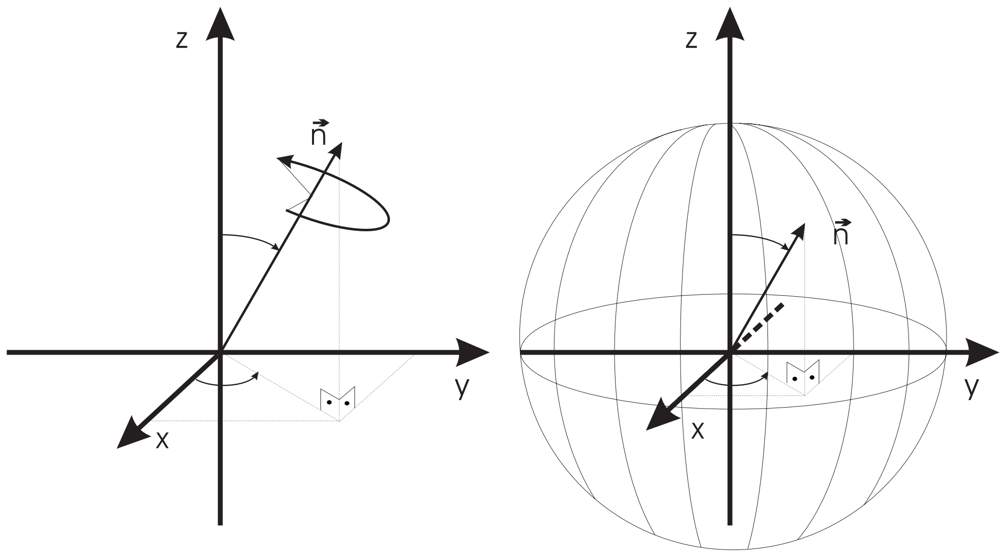

Figure 1: *A rotation can be characterized by an axis of rotation and an angle of rotation around that axis (left). The parameter space of an axis and angle parameterization (right): each rotation is associated with a three dimensional vector pointing in the direction* (θ, φ) *with magnitude equal to the rotation angle* ψ*.*

Because of the redundancy expressed in Eq. (104), two points on the surface of the sphere on opposite ends of a diameter are equivalent to each other [4] (see also Fig. 4).

Having the cartesian coordinates of the axis of rotation it is easy to get its spherical coordinates. We have

$$\hat{n}_x = \sin\theta\cos\varphi, \tag{105}$$

$$\hat{n}_y = \sin\theta\sin\varphi, \tag{106}$$

$$\hat{n}_z = \cos\theta. \tag{107}$$

From (107) we get

$$\theta = \arccos\hat{n}_z. \tag{108}$$

Since, due to (101), $\theta \in [0, \pi]$

$$\sin\theta \geq 0. \tag{109}$$

Therefore, following to $\sin^2\theta + \cos^2\theta = 1$ we have

$$\sin\theta = +\sqrt{1 - \cos^2\theta} \tag{110}$$

$$\stackrel{(107)}{=} \quad +\sqrt{1-\hat{n}_z^2}. \tag{111}$$

In this way by combining (105) and (111) we get

$$\varphi = \arccos \frac{\hat{n}_x}{\sqrt{1-\hat{n}_z^2}}. \tag{112}$$

Due to (12) the computer's result of (112) will be in the range between 0 and π. But following to (102) φ should be in the range between 0 and 2π. Therefore in addition to (105) we have to recognize (106). From (106) we get

$$\sin\varphi = \frac{\hat{n}_y}{\sin\theta}. \tag{113}$$

Since, due to (109) $\sin\theta \geq 0$, the sign of $\sin\varphi$ and therefore the range of φ completely depends on $\hat{n}_y$: if $\hat{n}_y \geq 0$, $\sin\varphi \geq 0$ so that φ is in the range between 0 and π; if $\hat{n}_y < 0$, $\sin\varphi < 0$ so that φ is in the range between π and 2π. Thus we have

$$\hat{n}_y \geq 0 \qquad \Longrightarrow \qquad \varphi = \arccos \frac{\hat{n}_x}{\sqrt{1-\hat{n}_z^2}}, \tag{114}$$

$$\hat{n}_y < 0 \qquad \Longrightarrow \qquad \varphi = 2\pi - \arccos \frac{\hat{n}_x}{\sqrt{1-\hat{n}_z^2}}. \tag{115}$$

Again (115) does not change the validity of (105) due to (39).

The essence of (108), (114) and (115) can be combined in the following pseudo-code:

$\theta = \arccos \hat{n}_z$;
$\varphi = \arccos \frac{\hat{n}_x}{\sqrt{1-\hat{n}_z^2}}$;
IF $(\hat{n}_y < 0)$ THEN $\varphi \leftarrow 2\pi - \varphi$;

All we have to do is to determine the axis of rotation in cartesian coordinates $\hat{\boldsymbol{n}}$ and the angle of rotation ψ.

4.1 Determination of the angle of rotation

To determine the angle of rotation we have to find an expression for an arbitrary rotation matrix in terms of the axis $\hat{\boldsymbol{n}}$ and the angle of rotation ψ. Let us assume that we have already determined the axis of rotation $\hat{\boldsymbol{n}}$ (we will perform this step in the next subsection with the results from this subsection). Then the rotation around $\hat{\boldsymbol{n}}$ can be decomposed into three steps, e.g. into

1. Rotate the coordinate system so that the x-axis coincides with the rotation axis described by $\hat{\boldsymbol{n}}$.

2. Rotate about ψ around the x-axis.

3. Rotate the coordinate system back so that the x-axis is in its old position.

The advantage of such a decomposition is that it is easy to specify the rotation matrices belonging to each of these steps. The easiest step is the rotation about ψ around the x-axis

$$\boldsymbol{R}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}. \tag{116}$$

If we denote the rotation matrix that rotates the x-axis to the rotation axis $\hat{\boldsymbol{n}}$ by $\tilde{\boldsymbol{R}}$ we have for the rotation matrix $\boldsymbol{R}_{\hat{n}}$ describing the rotation around $\hat{\boldsymbol{n}}$

$$\boldsymbol{R}_{\hat{n}} = \tilde{\boldsymbol{R}}\boldsymbol{R}_x\tilde{\boldsymbol{R}}^t \tag{117}$$

by recognizing $\tilde{\boldsymbol{R}}^{-1} = \tilde{\boldsymbol{R}}^t$. Note that the first performed rotation is the *inverse* of a rotation of the x-axis to the rotation axis $\hat{\boldsymbol{n}}$ which is the same as rotating the coordinate system in a way so that the x-axis coincides with $\hat{\boldsymbol{n}}$. It is now our task to construct $\tilde{\boldsymbol{R}}$.

Since $\tilde{\boldsymbol{R}}$ is defined by rotating the x-axis to the rotation axis described by $\hat{\boldsymbol{n}}$, we have

$$\tilde{\boldsymbol{R}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{n}_x \\ \hat{n}_y \\ \hat{n}_z \end{pmatrix}. \tag{118}$$

In addition we know that the image $\hat{\boldsymbol{n}}_\perp$ of $(0,1,0)^t$ must be perpendicular to $(\hat{n}_x, \hat{n}_y, \hat{n}_z)$ since angles are invariant under rotations

$$\begin{pmatrix} \hat{n}_{\perp_x} \\ \hat{n}_{\perp_y} \\ \hat{n}_{\perp_z} \end{pmatrix} \cdot \begin{pmatrix} \hat{n}_x \\ \hat{n}_y \\ \hat{n}_z \end{pmatrix} = 0. \tag{119}$$

$\hat{\boldsymbol{n}}_\perp$ is not uniquely determined by (119). One possible solution for $\hat{\boldsymbol{n}}_\perp$ from (119) is

$$\begin{pmatrix} -\hat{n}_y - \hat{n}_z \\ \hat{n}_x \\ \hat{n}_x \end{pmatrix}. \tag{120}$$

Since lengths are preserved under rotations the image $\hat{\boldsymbol{n}}_\perp$ of $(0,1,0)^t$ in addition has to be normalized to 1:

$$\begin{pmatrix} \hat{n}_{\perp_x} \\ \hat{n}_{\perp_y} \\ \hat{n}_{\perp_z} \end{pmatrix} = \frac{1}{\sqrt{(-\hat{n}_y-\hat{n}_z)^2+\hat{n}_x^2+\hat{n}_x^2}} \begin{pmatrix} -\hat{n}_y-\hat{n}_z \\ \hat{n}_x \\ \hat{n}_x \end{pmatrix} = \frac{1}{\sqrt{2+2\hat{n}_y\hat{n}_z-\hat{n}_y^2-\hat{n}_z^2}} \begin{pmatrix} -\hat{n}_y-\hat{n}_z \\ \hat{n}_x \\ \hat{n}_x \end{pmatrix} \tag{121}$$

where in the last equation we have made use of the normalization condition for $\hat{\boldsymbol{n}}$, $\hat{n}_x^2+\hat{n}_y^2+\hat{n}_z^2=1$. Since

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{122}$$

and since the cross product is invariant under rotations, the image $\hat{\boldsymbol{n}}_\times$ of the third base vector $(0,0,1)^t$ is

$$\begin{aligned} \hat{\boldsymbol{n}}_\times &= \hat{\boldsymbol{n}} \times \hat{\boldsymbol{n}}_\perp \\ &\overset{(118),(121)}{=} \begin{pmatrix} \hat{n}_x \\ \hat{n}_y \\ \hat{n}_z \end{pmatrix} \times \frac{1}{\sqrt{2+2\hat{n}_y\hat{n}_z-\hat{n}_y^2-\hat{n}_z^2}} \begin{pmatrix} -\hat{n}_y-\hat{n}_z \\ \hat{n}_x \\ \hat{n}_x \end{pmatrix} \\ &= \frac{1}{\sqrt{2+2\hat{n}_y\hat{n}_z-\hat{n}_y^2-\hat{n}_z^2}} \begin{pmatrix} \hat{n}_y\hat{n}_x-\hat{n}_z\hat{n}_x \\ \hat{n}_y^2-\hat{n}_y\hat{n}_z-1 \\ -\hat{n}_z^2+\hat{n}_y\hat{n}_z+1 \end{pmatrix} \end{aligned} \tag{123}$$

where in the last equation we again have made use of the normalization condition for $\hat{\boldsymbol{n}}$.

Since the columns of a matrix are the images of the base vectors we get from (118), (121) and (123) for $\tilde{\boldsymbol{R}}$

$$\tilde{\boldsymbol{R}} = \begin{pmatrix} \hat{n}_x & \frac{-\hat{n}_y-\hat{n}_z}{\sqrt{2+2\hat{n}_y\hat{n}_z-\hat{n}_y^2-\hat{n}_z^2}} & \frac{\hat{n}_y\hat{n}_x-\hat{n}_z\hat{n}_x}{\sqrt{2+2\hat{n}_y\hat{n}_z-\hat{n}_y^2-\hat{n}_z^2}} \\ \hat{n}_y & \frac{\hat{n}_x}{\sqrt{2+2\hat{n}_y\hat{n}_z-\hat{n}_y^2-\hat{n}_z^2}} & \frac{\hat{n}_y^2-\hat{n}_y\hat{n}_z-1}{\sqrt{2+2\hat{n}_y\hat{n}_z-\hat{n}_y^2-\hat{n}_z^2}} \\ \hat{n}_z & \frac{\hat{n}_x}{\sqrt{2+2\hat{n}_y\hat{n}_z-\hat{n}_y^2-\hat{n}_z^2}} & \frac{-\hat{n}_z^2+\hat{n}_y\hat{n}_z+1}{\sqrt{2+2\hat{n}_y\hat{n}_z-\hat{n}_y^2-\hat{n}_z^2}} \end{pmatrix}. \tag{124}$$

It is now straightforward to calculate $\boldsymbol{R}_{\hat{n}} = \tilde{\boldsymbol{R}}\boldsymbol{R}_x\tilde{\boldsymbol{R}}^t$ from (117) by using (116) and (124). However it is advantageous to write down $\tilde{\boldsymbol{R}}\boldsymbol{R}_x\tilde{\boldsymbol{R}}^t$ at first with a general matrix $\tilde{\boldsymbol{R}}$ to facilitate the calculations:

$$\begin{aligned} \boldsymbol{R}_{\hat{n}} &= \begin{pmatrix} \tilde{r}_{00} & \tilde{r}_{01} & \tilde{r}_{02} \\ \tilde{r}_{10} & \tilde{r}_{11} & \tilde{r}_{12} \\ \tilde{r}_{20} & \tilde{r}_{21} & \tilde{r}_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \tilde{r}_{00} & \tilde{r}_{10} & \tilde{r}_{20} \\ \tilde{r}_{01} & \tilde{r}_{11} & \tilde{r}_{21} \\ \tilde{r}_{02} & \tilde{r}_{12} & \tilde{r}_{22} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{r}_{00}^2+\left(\tilde{r}_{01}^2+\tilde{r}_{02}^2\right)\cos\psi & \begin{matrix}\tilde{r}_{00}\tilde{r}_{10}+\left(\tilde{r}_{01}\tilde{r}_{11}+\tilde{r}_{02}\tilde{r}_{12}\right)\cos\psi- \\ \left(\tilde{r}_{01}\tilde{r}_{12}-\tilde{r}_{02}\tilde{r}_{11}\right)\sin\psi\end{matrix} & \begin{matrix}\tilde{r}_{00}\tilde{r}_{20}+\left(\tilde{r}_{01}\tilde{r}_{21}+\tilde{r}_{02}\tilde{r}_{22}\right)\cos\psi+ \\ \left(\tilde{r}_{02}\tilde{r}_{21}-\tilde{r}_{01}\tilde{r}_{22}\right)\sin\psi\end{matrix} \\ \begin{matrix}\tilde{r}_{00}\tilde{r}_{10}+\left(\tilde{r}_{01}\tilde{r}_{11}+\tilde{r}_{02}\tilde{r}_{12}\right)\cos\psi+ \\ \left(\tilde{r}_{01}\tilde{r}_{12}-\tilde{r}_{02}\tilde{r}_{11}\right)\sin\psi\end{matrix} & \tilde{r}_{10}^2+\left(\tilde{r}_{11}^2+\tilde{r}_{12}^2\right)\cos\psi & \begin{matrix}\tilde{r}_{10}\tilde{r}_{20}+\left(\tilde{r}_{11}\tilde{r}_{21}+\tilde{r}_{12}\tilde{r}_{22}\right)\cos\psi- \\ \left(\tilde{r}_{11}\tilde{r}_{22}-\tilde{r}_{12}\tilde{r}_{21}\right)\sin\psi\end{matrix} \\ \begin{matrix}\tilde{r}_{00}\tilde{r}_{20}+\left(\tilde{r}_{01}\tilde{r}_{21}+\tilde{r}_{02}\tilde{r}_{22}\right)\cos\psi- \\ \left(\tilde{r}_{02}\tilde{r}_{21}-\tilde{r}_{01}\tilde{r}_{22}\right)\sin\psi\end{matrix} & \begin{matrix}\tilde{r}_{10}\tilde{r}_{20}+\left(\tilde{r}_{11}\tilde{r}_{21}+\tilde{r}_{12}\tilde{r}_{22}\right)\cos\psi+ \\ \left(\tilde{r}_{11}\tilde{r}_{22}-\tilde{r}_{12}\tilde{r}_{21}\right)\sin\psi\end{matrix} & \tilde{r}_{20}^2+\left(\tilde{r}_{21}^2+\tilde{r}_{22}^2\right)\cos\psi \end{pmatrix}. \end{aligned} \tag{125}$$

Now we use (124) in (125) to get for $\boldsymbol{R}_{\hat{n}}$

$$\boldsymbol{R}_{\hat{n}} = \begin{pmatrix} \hat{n}_x^2+\left(1-\hat{n}_x^2\right)\cos\psi & \hat{n}_x\hat{n}_y(1-\cos\psi)-\hat{n}_z\sin\psi & \hat{n}_x\hat{n}_z(1-\cos\psi)+\hat{n}_y\sin\psi \\ \hat{n}_x\hat{n}_y(1-\cos\psi)+\hat{n}_z\sin\psi & \hat{n}_y^2+\left(1-\hat{n}_y^2\right)\cos\psi & \hat{n}_y\hat{n}_z(1-\cos\psi)-\hat{n}_x\sin\psi \\ \hat{n}_x\hat{n}_z(1-\cos\psi)-\hat{n}_y\sin\psi & \hat{n}_y\hat{n}_z(1-\cos\psi)+\hat{n}_x\sin\psi & \hat{n}_z^2+\left(1-\hat{n}_z^2\right)\cos\psi \end{pmatrix}. \tag{126}$$

It was our aim to determine the angle of rotation ψ from a given rotation matrix $\boldsymbol{R}$. Therefore a general rotation matrix $\boldsymbol{R}$ like in (5) should be compared with $\boldsymbol{R}_{\hat{n}}$ from (126). We get e.g.

$$r_{00} = \hat{n}_x^2 + \left(1-\hat{n}_x^2\right)\cos\psi. \tag{127}$$

Thus, if $\hat{n}_x \neq 1$ we have

$$\psi = \arccos\left(\frac{r_{00}-\hat{n}_x^2}{1-\hat{n}_x^2}\right). \tag{128}$$

If $\hat{n}_x = 1$ we conclude from the normalization of $\hat{\boldsymbol{n}}$, $\hat{n}_x^2+\hat{n}_y^2+\hat{n}_z^2 = 1$, that $\hat{n}_y = \hat{n}_z = 0$. Inserting $\hat{\boldsymbol{n}} = (1,0,0)^t =: \hat{\boldsymbol{n}}_x$ in $\boldsymbol{R}_{\hat{n}}$ from (126) we get

$$\boldsymbol{R}_{\hat{n}_x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix} \tag{129}$$

so that ψ can be determined e.g. by

$$\psi = \arccos r_{11}. \tag{130}$$

The essence of (128) and (130) can be combined into the following pseudo-code:

IF $(\hat{n}_x \neq 1)$ THEN $\psi = \arccos\left(\frac{r_{00}-\hat{n}_x^2}{1-\hat{n}_x^2}\right)$;
ELSE $\psi = \arccos r_{11}$;

4.2 Determination of the axis of rotation

We want to determine the axis of rotation from a given rotation matrix $\boldsymbol{R}$. The characteristic of the axis of rotation is that it is invariant under the rotation. In other words it is the eigenvector of the corresponding rotation matrix to the eigenvalue 1

$$\begin{aligned} \boldsymbol{R}\hat{\boldsymbol{n}} &= \begin{pmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{pmatrix} \begin{pmatrix} \hat{n}_x \\ \hat{n}_y \\ \hat{n}_z \end{pmatrix} \\ &\overset{!}{=} 1 \begin{pmatrix} \hat{n}_x \\ \hat{n}_y \\ \hat{n}_z \end{pmatrix}. \end{aligned} \tag{131}$$

The most elegant way to determine $\hat{\boldsymbol{n}}$ is by recognizing that $\boldsymbol{R}^t$ has the same eigenvector and eigenvalue as $\boldsymbol{R}$ since

$$\begin{aligned} 1\hat{\boldsymbol{n}} &= \boldsymbol{1}\,\hat{\boldsymbol{n}} = \left(\boldsymbol{R}^t\boldsymbol{R}\right)\hat{\boldsymbol{n}} = \boldsymbol{R}^t\left(\boldsymbol{R}\hat{\boldsymbol{n}}\right) \\ &\overset{(131)}{=} \boldsymbol{R}^t\hat{\boldsymbol{n}}. \end{aligned} \tag{132}$$

Combining (131) and (132) we have

$$\left(\boldsymbol{R}-\boldsymbol{R}^t\right)\hat{\boldsymbol{n}} = \begin{pmatrix} 0 & r_{01}-r_{10} & r_{02}-r_{20} \\ r_{10}-r_{10} & 0 & r_{12}-r_{21} \\ r_{20}-r_{02} & r_{21}-r_{12} & 0 \end{pmatrix} \begin{pmatrix} \hat{n}_x \\ \hat{n}_y \\ \hat{n}_z \end{pmatrix} = 0. \tag{133}$$

We already know that only 2 of the 3 equations in (133) are independent, since, if $\hat{\boldsymbol{n}}$ is an eigenvector of $\boldsymbol{R}$, also all multiples of $\hat{\boldsymbol{n}}$ are eigenvectors of $\boldsymbol{R}$ to the same eigenvalue. Considering the equations corresponding to the first 2 rows in (133)

$$(r_{01}-r_{10})\,\hat{n}_y + (r_{02}-r_{20})\,\hat{n}_z = 0, \tag{134}$$

$$(r_{10}-r_{01})\,\hat{n}_x + (r_{12}-r_{21})\,\hat{n}_z = 0 \tag{135}$$

we get

$$\hat{n}_y = \frac{r_{20}-r_{02}}{r_{01}-r_{10}}\hat{n}_z, \tag{136}$$

$$\hat{n}_x = \frac{r_{12}-r_{21}}{r_{01}-r_{10}}\hat{n}_z \tag{137}$$

provided that $r_{01} - r_{10} \neq 0$. If $r_{01} - r_{10} = 0$, (134) and (135) reduce to

$$\begin{aligned}
(r_{02} - r_{20})\,\hat{n}_z &= 0, && (138)\\
(r_{12} - r_{21})\,\hat{n}_z &= 0 && (139)
\end{aligned}$$

so that we have to consider the third equation following from (133)

$$(r_{20} - r_{02})\,\hat{n}_x + (r_{21} - r_{12})\,\hat{n}_y = 0. \quad (140)$$

If $r_{20} \neq r_{02}$ we get from (138) and (140)

$$\begin{aligned}
\hat{n}_z &= 0, && (141)\\
\hat{n}_x &= \frac{r_{12} - r_{21}}{r_{20} - r_{02}}\hat{n}_y. && (142)
\end{aligned}$$

Otherwise, if $r_{20} = r_{02}$, (140) reduces to

$$(r_{21} - r_{12})\,\hat{n}_y = 0, \quad (143)$$

so that we can conclude from (139) and (143)

$$\begin{aligned}
\hat{n}_z &= 0, && (144)\\
\hat{n}_y &= 0 && (145)
\end{aligned}$$

if $r_{21} \neq r_{12}$. If in addition $r_{21} = r_{12}$, we have $\boldsymbol{R} = \boldsymbol{R}^t \stackrel{!}{=} \boldsymbol{R}^{-1}$, so that $\boldsymbol{R}^2 = \boldsymbol{1}$. In other words, rotating two times around the same axis about the same angle will give the identity. So we can conclude that we rotate about an angle of 0° or 180° around an unknown axis of rotation. If we rotate about 0° already $\boldsymbol{R}$ (and not just $\boldsymbol{R}^2$) is the identity ($\boldsymbol{R} = \boldsymbol{1}$). Thus, all vectors $\hat{\boldsymbol{n}}$ are eigenvectors of $\boldsymbol{R}$. In such a case we choose the same eigenvector as in the case $r_{21} \neq r_{12}$. If we rotate about $\psi = 180°$ we get from (126) with $\sin\psi = 0$ and $\cos\psi = -1$

$$\boldsymbol{R}_{\hat{n}} = \begin{pmatrix} 2\hat{n}_x^2 - 1 & 2\hat{n}_x\hat{n}_y & 2\hat{n}_x\hat{n}_z \\ 2\hat{n}_x\hat{n}_y & 2\hat{n}_y^2 - 1 & 2\hat{n}_y\hat{n}_z \\ 2\hat{n}_x\hat{n}_z & 2\hat{n}_y\hat{n}_z & 2\hat{n}_z^2 - 1 \end{pmatrix}. \quad (146)$$

If we compare this with a given rotation matrix $\boldsymbol{R}$ as in (131) we can conclude from the following two resulting equations whether the angle of rotation ψ is 0° or 180°,

$$\begin{aligned}
r_{00} &= 2\hat{n}_x^2 - 1, && (147)\\
r_{11} &= 2\hat{n}_y^2 - 1. && (148)
\end{aligned}$$

If $r_{00} = 1$ and $r_{11} = 1$ we can conclude that $\psi = 0$ (i.e. $\boldsymbol{R} = \boldsymbol{1}$), since otherwise we would get in this case from (147) that $\hat{n}_x^2 = 1$ and at the same time from (148) that $\hat{n}_y^2 = 1$. But since $\hat{\boldsymbol{n}}$ is normalized in (146) (see (118) for its precise definition) $\hat{n}_x^2 = \hat{n}_y^2 = 1$ is not possible.

To determine $\hat{\boldsymbol{n}}$ in the case $\psi = 180°$ we get from (147), (148) and (146)

$$\hat{n}_x = \pm\sqrt{\frac{r_{00}+1}{2}}, \tag{149}$$

$$\hat{n}_y = \pm\sqrt{\frac{r_{11}+1}{2}}, \tag{150}$$

$$\hat{n}_z = \pm\sqrt{\frac{r_{22}+1}{2}}. \tag{151}$$

To get the correct signs for $\hat{n}_x, \hat{n}_y$ and $\hat{n}_z$ we have to compare again (146) with the given rotation matrix $\boldsymbol{R}$,

$$r_{01} = 2\hat{n}_x\hat{n}_y, \tag{152}$$

$$r_{02} = 2\hat{n}_x\hat{n}_z, \tag{153}$$

$$r_{12} = 2\hat{n}_y\hat{n}_z. \tag{154}$$

Due to the redundancy (104) it is possible to arbitrarily choose one of the signs in (149–151), e.g.

$$\hat{n}_z = +\sqrt{\frac{r_{22}+1}{2}}. \tag{155}$$

Then we can directly follow from (153) and (154) the signs of $\hat{n}_x$ and $\hat{n}_y$ if $\hat{n}_z \neq 0$,

$$r_{02} < 0 \quad \overset{(153),(155)}{\Longrightarrow} \quad \hat{n}_x < 0, \tag{156}$$

$$r_{12} < 0 \quad \overset{(154),(155)}{\Longrightarrow} \quad \hat{n}_y < 0. \tag{157}$$

If $\hat{n}_z = 0$ we again have (due to the redundancy (104)) the freedom to arbitrarily choose one of the signs of $\hat{n}_x$ or $\hat{n}_y$, e.g.

$$\hat{n}_x = +\sqrt{\frac{r_{00}+1}{2}}. \tag{158}$$

Then, for the sign of $\hat{n}_y$ we have to consider (152),

$$r_{01} < 0 \overset{(152),(158)}{\Longrightarrow} \hat{n}_y < 0. \tag{159}$$

If in addition $\hat{n}_x = 0$ we can arbitrarily choose the sign of $\hat{n}_y$, e.g.

$$\hat{n}_y = -\sqrt{\frac{r_{11}+1}{2}}. \tag{160}$$

The essence of (136), (137), (141), (142), (144), (145), (147), (148), (149), (150), (155), (156), (157) and (159) can be combined into the following pseudo-code where we additionally take the normalization of $\hat{\boldsymbol{n}}$ into account:

IF $(r_{01} \neq r_{10})$ THEN
{
$\hat{n}_y = \frac{r_{20}-r_{02}}{r_{01}-r_{10}}$;
$\hat{n}_x = \frac{r_{12}-r_{21}}{r_{01}-r_{10}}$;
$r = \sqrt{\hat{n}_x^2 + \hat{n}_y^2 + 1}$;
$\hat{n}_z = \frac{1}{r}$;
$\hat{n}_y \leftarrow \frac{\hat{n}_y}{r}$;
$\hat{n}_x \leftarrow \frac{\hat{n}_x}{r}$;
}
ELSE IF $(r_{20} \neq r_{02})$ THEN

{
$\hat{n}_z = 0$;
$\hat{n}_x = \frac{r_{12}-r_{21}}{r_{20}-r_{02}}$;
$r = \sqrt{\hat{n}_x^2 + 1}$;
$\hat{n}_y = \frac{1}{r}$;
$\hat{n}_x \leftarrow \frac{\hat{n}_x}{r}$;
}
ELSE IF $(r_{21} \neq r_{12} \quad$ OR $\quad (r_{00} = 1 \quad$ AND $\quad r_{11} = 1))$ THEN
{
$\hat{n}_z = 0$;
$\hat{n}_y = 0$;
$\hat{n}_x = 1$;
}
ELSE
{
$\hat{n}_x = \sqrt{\frac{r_{00}+1}{2}}$;
$\hat{n}_y = \sqrt{\frac{r_{11}+1}{2}}$;

```
    n̂_z = sqrt((r_22+1)/2);
    IF (n̂_z ≠ 0) THEN
    {
        IF (r_02 < 0) THEN n̂_x ← −n̂_x;
        IF (r_12 < 0) THEN n̂_y ← −n̂_y;
    }
    ELSE
    {
        IF (r_01 < 0) THEN n̂_y ← −n̂_y;
    }
}
```

We will discuss the neighborhood structure of the axis and angle parameterization in further detail in a comparison with the quaternion parameterization described next.

5 Quaternion neighborhood structure

The quaternion parameterization is very similar to an axis and angle parameterization in its neighborhood structure. But before diving in the quaternion neighborhood structure we give at first a short introduction to quaternions and their calculation rules, and we describe how to extract the quaternion parameters from a general rotation matrix.

5.1 Introduction to quaternions

A quaternion $\dot{q}$ can be represented in the complex number notation

$$\dot{q} = q_0 + iq_x + jq_y + kq_z \tag{161}$$

with real part q_0 and three imaginary parts q_x, q_y, q_z. For the imaginary units i, j, k the following equations hold:

$$\begin{array}{lllllllll} i^2 & = & -1, & j^2 & = & -1 & k^2 & = & -1, \\ ij & = & k, & jk & = & i, & ki & = & j, \\ ji & = & -k, & kj & = & -i, & ik & = & -j. \end{array} \tag{162}$$

With (162) the multiplication of quaternions $\dot{r}$ and $\dot{q}$ can be defined in terms of the products of their components,

$$\begin{aligned}\dot{r}\dot{q} &= (r_0q_0 - r_xq_x - r_yq_y - r_zq_z) \\ &\quad + i\,(r_0q_x + r_xq_0 + r_yq_z - r_zq_y) \\ &\quad + j\,(r_0q_y - r_xq_z + r_yq_0 + r_zq_x) \\ &\quad + k\,(r_0q_z + r_xq_y - r_yq_x + r_zq_0)\,.\end{aligned}$$

In general $\dot{r}\dot{q} \neq \dot{q}\dot{r}$.

The dot product of two quaternions is the sum of products of corresponding components:

$$\dot{p}\cdot\dot{q} = p_0q_0 + p_xq_x + p_yq_y + p_zq_z. \tag{163}$$

The square of the magnitude of a quaternion is the dot product of the quaternion with itself:

$$||\dot{q}||^2 = \dot{q}\cdot\dot{q}. \tag{164}$$

A unit quaternion is a quaternion whose magnitude equals 1.

The conjugate of a quaternion negates its imaginary parts:

$$\dot{q}^* = q_0 - iq_x - jq_y - kq_z. \tag{165}$$

Vectors can be represented by purely imaginary quaternions. If $\boldsymbol{r} = (x,y,z)^T$, we can use the quaternion

$$\dot{r} = 0 + ix + jy + kz. \tag{166}$$

Scalars can be similarly represented by using real quaternions.

Using the fact that only rotations preserve dot products and cross products, we can represent a rotation by a quaternion if we can find a way of mapping purely imaginary quaternions (that represent vectors) into purely imaginary quaternions in such a way that dot and cross products are preserved. It can be shown that the composite product

$$\dot{r}' = \dot{q}\dot{r}\dot{q}^*, \tag{167}$$

where $\dot{q}$ is a unit quaternion, transforms the imaginary quaternion $\dot{r}$ into an imaginary quaternion $\dot{r}'$ and preserves the dot and cross products between $\dot{r}$ and a second imaginary quaternion $\dot{r}_2$. Since

$$(-\dot{q})\,\dot{r}\,(-\dot{q}^*) = \dot{q}\dot{r}\dot{q}^* \tag{168}$$

$-\dot{q}$ represents the same rotation as $\dot{q}$.

It is straightforward to verify that the composition of rotations corresponds to multiplication of quaternions:

$$\begin{aligned} \dot{r}'' &= \dot{p}\dot{r}'\dot{p}^* \\ &= \dot{p}\left(\dot{q}\dot{r}\dot{q}^*\right)\dot{p}^* \\ &\overset{\cdots}{=} \left(\dot{p}\dot{q}\right)\dot{r}\left(\dot{p}\dot{q}\right)^* . \end{aligned}$$

The overall rotation is represented by the unit quaternion $\dot{p}\dot{q}$.

It may be of interest to note that it takes fewer arithmetic operations to multiply two quaternions than it does to multiply two 3×3 matrices. Also, since calculations are not carried out with infinite precision on a computer the product of many orthonormal matrices may no longer be orthonormal, just as the product of many unit quaternions may no longer be a unit quaternion. However it is trivial to find the nearest unit quaternion, whereas it is quite difficult to find the nearest orthonormal matrix.

Unit quaternions are closely related to the geometrically intuitive axis and angle notation. A rotation by an angle ψ about the axis defined by the unit vector $\hat{\boldsymbol{n}} = (\hat{n}_x, \hat{n}_y, \hat{n}_z)^T$ can be represented by the unit quaternion

$$\dot{q} = \cos\frac{\psi}{2} + \sin\frac{\psi}{2}\left(i\hat{n}_x + j\hat{n}_y + k\hat{n}_z\right). \tag{169}$$

The relation of a unit quaternion $\dot{q}$ to the familiar orthonormal rotation matrix $\boldsymbol{R}_{\dot{q}}$ is given by

$$\boldsymbol{R}_{\dot{q}} = \begin{pmatrix} \left(q_0^2+q_x^2-q_y^2-q_z^2\right) & 2\left(q_xq_y-q_0q_z\right) & 2\left(q_xq_z+q_0q_y\right) \\ 2\left(q_yq_x+q_0q_z\right) & \left(q_0^2-q_x^2+q_y^2-q_z^2\right) & 2\left(q_yq_z-q_0q_x\right) \\ 2\left(q_zq_x-q_0q_y\right) & 2\left(q_zq_y+q_0q_x\right) & \left(q_0^2-q_x^2-q_y^2+q_z^2\right) \end{pmatrix}. \tag{170}$$

5.2 Extraction of quaternion parameters from rotation matrices

We use the relation (169) to extract quaternion parameters from a rotation matrix. From (169) we get,

$$q_0 = \cos\frac{\psi}{2}, \tag{171}$$

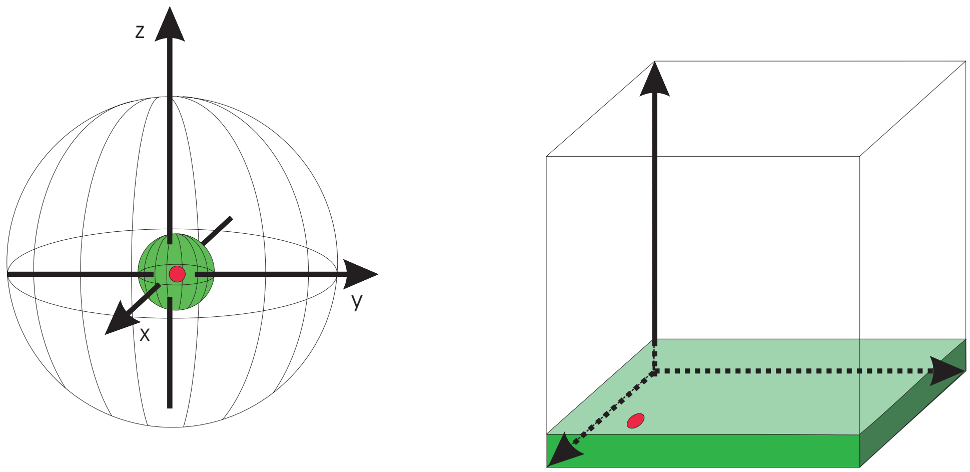

Figure 2: *The neighborhood of a rotation near the identity transformation in the axis and angle parameterization (left). Since the spherical coordinates* (θ, φ) *of points in the neighborhood of the origin take on all values in their definition range, the neighborhood of the identity transformation is spread over the whole plane in the Hough table belonging to an angle of rotation* $\psi \approx 0$ *(right).*

$$q_x = \sin\frac{\psi}{2}\hat{n}_x, \tag{172}$$

$$q_y = \sin\frac{\psi}{2}\hat{n}_y, \tag{173}$$

$$q_z = \sin\frac{\psi}{2}\hat{n}_z. \tag{174}$$

The parameters of the axis of rotation $\hat{n}_x, \hat{n}_y$ and $\hat{n}_z$ can be determined by the algorithm on page 30, the angle of rotation ψ can be determined by the algorithm on page 26.

Note that since $0 \leq \psi \leq \pi$ (cf. (103)) q_0 is always greater or equal to zero.

5.3 Comparison of neighborhoods in the quaternion and axis and angle parameterization

Although we use the same algorithms to determine quaternion parameters and the parameters in the axis and angle representation we will argue that the quaternion parameterization is more suited for the Hough table approach to registration as described in [2]. The reason for this can be best explained by a visualization of

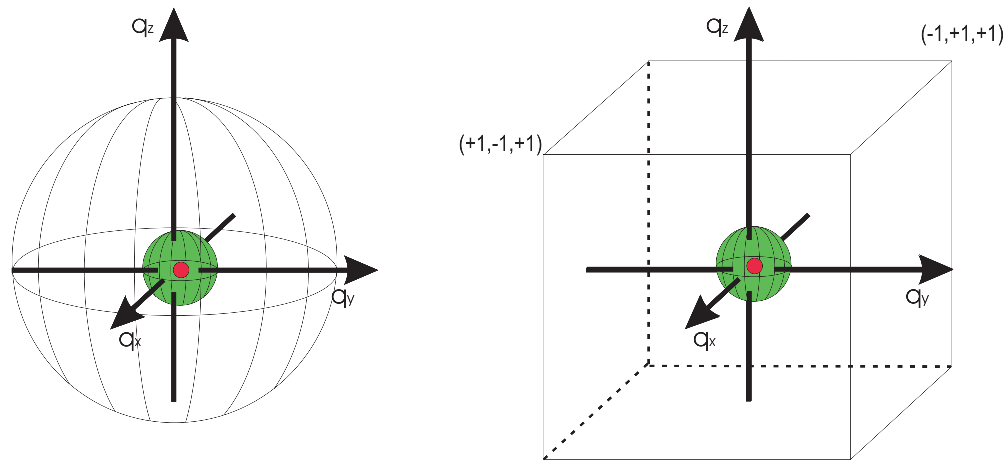

Figure 3: *The neighborhood of a rotation near the identity transformation in the quaternion parameterization with parameters q_x, q_y and q_z: (left) in the parameter space, (right) in the Hough table. Since there is no relevant difference between the representations in the parameter space and the Hough table (in contrast to the axis and angle parameterization) both representations can be combined (see also Fig. 5).*

the parameter space of an axis and angle parameterization (see Fig. 1). As already mentioned the structure of the axis and angle parameter space can be visualized by associating each rotation with a three dimensional vector $\boldsymbol{v} = \psi\hat{\boldsymbol{n}}$ pointing in the direction of the rotation axis with magnitude equal to the angle of rotation ψ. If we are now close to the origin, which means close to the identity transformation, the neighborhood of a rotation can be extended to an area around the origin. But if we use spherical coordinates, as it is done in the axis and angle parameterization, the points in this area take on all values in their definition range. Therefore, if we use the axis an angle parameters as indices in a Hough table the neighborhood of the identity transformation is spread over the whole plane in the Hough table belonging to an angle of rotation $\psi \approx 0$ (see Fig 2).

In the quaternion parameterization we use the cartesian coordinates of the points in the neighborhood of the origin. Since in cartesian coordinates the origin is no extraordinary point, we have not the ambiguities as in the axis and angle case (see Fig. 3).

As already mentioned in Sec. 4, because of the redundancy $\boldsymbol{R}_{-\hat{n}}(\pi) = \boldsymbol{R}_{\hat{n}}(\pi)$, two points in the axis and angle parameterization on the surface of the parameter-

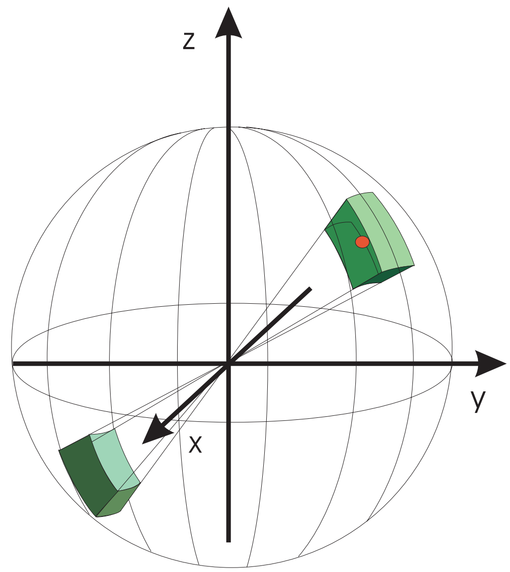

Figure 4: *The neighborhood structure of the axis and angle parameterization following from the redundancy* $\boldsymbol{R}_{-\hat{n}}(\pi) = \boldsymbol{R}_{\hat{n}}(\pi)$ *visualized in the parameter space.*

izing sphere on opposite ends of a diameter are equivalent to each other. From this fact it follows a somewhat involved neighborhood structure in the parameter space (see Fig. 4) that would also be present in a Hough table, i.e. when the parameters are used as indices in a 3-dimensional table: the neighborhood is divided into two separated areas in the parameter space. Since the quaternion parameterization is an axis and angle parameterization in cartesian coordinates, there is the same neighborhood structure in the quaternion parameterization (compare Fig. 5).

It is worth to mention that, since there is no relevant difference between the representations of quaternion parameters in the parameter space and in the Hough table, it is possible to combine both representations (see Fig. 3 and Fig. 5).

6 Final Conclusions

The best way to avoid problems with the determination of rotation parameters from a given rotation matrix is to choose a parameterization that has no principal problems in determining these parameters. The quaternion parameterization is such a parameterization. It has no problems for rotations near the identity transformation as it is the case for an axis and angle parameterization and there are

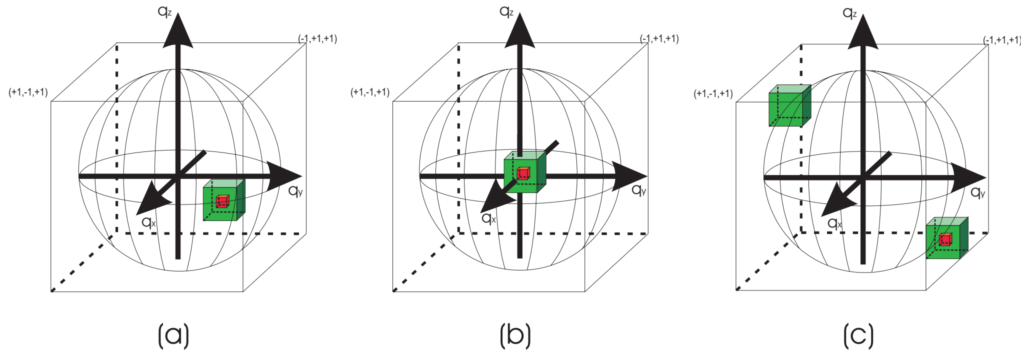

Figure 5: *The neighborhood structure of quaternion parameters* q_x, q_y, q_z *in the Hough table. Only Hough table cells in the sphere correspond to rotations. The neighborhood is visualized as a cube and not as a sphere since in the Hough table the neighborhood is given as a cube. (a) The usual case with trivial neighborhood structure. (b) In the vicinity of the identity transformation there is also a trivial neighborhood structure in contrast to the axis and angle parameterization. (c) In the vicinity of rotations by an angle* π *about an arbitrary axis of rotation the neighborhood structure is somewhat involved.*

also no problems for other parameter configurations as it is the case for Euler angle like parameterizations. Nevertheless the quaternion parameterization suffers from a complicated neighborhood in the vicinity of rotations about an rotation angle π.

Another drawback of the quaternion parameterization for the Hough table approach to registration as described in [2] is the high memory consumption of a Hough table using quaternion parameters: since only Hough table cells belonging to the sphere in Fig. 5 correspond to rotations, the Hough table as the bounding box of the parameter sphere needs nearly twice as much cells as are really needed for rotations because the overhead can be estimated as

$$(2r)^3 - \frac{4\pi}{3}r^3 = \left(8 - \frac{4\pi}{3}\right) r^3 = \left(2\frac{4\pi}{\pi} - \frac{4\pi}{3}\right) r^3 \stackrel{\pi\approx 3}{\approx} \frac{4\pi}{3} r^3. \tag{175}$$

Certainly, it is possible to avoid the memory overhead by some advanced rules which map cells inside the sphere to the computer memory but this would complicate the neighborhood of rotations even more and even for rotations that have a trivial neighborhood structure with the approach described till now.

Therefore in Subsec. 2.5 we proposed a compromise between the high memory consumption of a usual quaternion parameterization and the difficulty of a sophisticated map from permitted Hough cells to the computer memory: in the first iteration of the Hough table approach we suggest a quaternion parameterization with the disadvantage of a high memory consumption, respectively a slower convergence rate, since we have to choose a rougher Hough cell resolution because memory resources are a limiting factor. After detecting the correct Hough peak by an additional analysis of the neighborhood of each Hough cell we change for further iterations to an Euler angle like parameterization that has no parameterization problems in the vicinity of the detected Hough peak. Since the critical parameterizations in the Euler angle like parameterizations are far away from each other (see appendix A), such a parameterization can always be found. In this way we have a simple Hough table structure over all iterations of the algorithm. Only in the first iteration it is possible that the correct detection of the Hough peak can become a little bit involved if the Hough peak corresponds to a rotation near π about an arbitrary axis of rotation. But there is no real problem in determining the Hough peak even in this case.

A Relation between critical Euler parameter ranges

In Sec. 6 we claimed that the critical parameterizations in the Euler angle like parameterizations are far away from each other. We want to give a proof for this assertion.

In Subsec. 3.1 we demonstrated that the Euler angle parameters in the *zyx*-Euler case are essentially determined from a given rotation matrix as

$$\begin{aligned} \beta' &= \arcsin r_{02}, &(176)\\ \alpha' &= \arccos\left(\frac{r_{00}}{\cos\left(\arcsin r_{02}\right)}\right), &(177)\\ \gamma' &= \arccos\left(-\frac{r_{22}}{\cos\left(\arcsin r_{02}\right)}\right). &(178) \end{aligned}$$

Substituting in these equations the matrix entries r_{ij} by the matrix entries as given in the usual Euler angle parameterization (compare (1)) we get a relation between the parameters in the *zyx*- and the *zyz*-Euler case,

$$\beta' = \arcsin\left(\cos\alpha\sin\beta\right), \qquad (179)$$

$$\alpha' = \arccos\left(\frac{\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma}{\cos[\arcsin(\cos\alpha\sin\beta)]}\right), \tag{180}$$

$$\gamma' = \arccos\left(-\frac{\cos\beta}{\cos[\arcsin(\cos\alpha\sin\beta)]}\right). \tag{181}$$

The critical range for β' begins when it becomes difficult to determine α' and γ' due to a denominator in the arccos function that becomes close to zero. Therefore if

$$\cos[\arcsin(\cos\alpha\sin\beta)] \approx 0 \tag{182}$$

$$\Leftrightarrow \quad \arcsin(\cos\alpha\sin\beta) \approx \frac{\pi}{2}, \frac{3\pi}{2} \tag{183}$$

$$\Leftrightarrow \quad \cos\alpha\sin\beta \approx 1, -1 \tag{184}$$

$$\Leftrightarrow \quad \cos\alpha\sin\beta \approx |1| \tag{185}$$

we run into problems with β'. To become close to $|1|$ with $\cos\alpha\sin\beta$, α has to be near 0 or near π. Let us assume the worst case scenario[6] $\alpha = 0$ or $\alpha = \pi$. In this case we get from (179) for the relation between β' and β

$$\beta' = \pm\beta. \tag{186}$$

Therefore if we take for the critical ranges of β the ranges $[0^\circ, 45^\circ]$ and $[180^\circ - 45^\circ, 180^\circ]$ and if we take for the critical ranges of β' the ranges $[-90^\circ, -90^\circ + 45^\circ]$ and $[90^\circ - 45^\circ, 90^\circ]$, then we can be sure that if β is in one of its critical ranges, β' will be not.

References

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[6] Worst case scenario since it enlarges the critical range of β at most.